MSC 94A60

ENCRYPTION DEMANDS WITH KOLMOGOROV ALGORITHMIC METRIC

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The authors propose more general encryption demand than well-known avalanche effect: each two ciphertexts (including two encryptions of same plaintext) are to differ sufficiently. To formalize this demand the authors propose to use asymptotical minimal volume of algorithm of revertible transformation of one text into another one with same information.

Keywords: encryption, avalanche effect, Kolmogorov algorithmic metrics.

Авторлор шифрлөөдөн белгилүү кар көчкү эффектинен жалпыраак төмөнкү талап кылууну сунуштайт. Ар бир эки шифрленген билдирүү (анын ичинде ошол эле билдирүүну шифрлөө) бири-биринен олуттуу айырмалануу керек. Бул талапты формалдаштыруу үчүн авторлор билдирүүнү ошол эле маалыматты кармаган башка билдирүүгө кайтылуучу өзгөртүүчү алгоритмдин минималдуу көлөмүн колдонулат.

Урунттуу сөздөр: шифрлөө, кар көчкү эффекти, Колмогоров алгоритмдик метрикасы.

Авторы предлагают более общие требования к шифрованию, чем известный «лавинный эффект»: любые два зашифрованных текста (включая два шифрования одного и того же текста) должны существенно различаться. Для формализации этого требования авторы предлагают использовать асимптотически минимальный объем алгоритма обратимого преобразования одного текста в другой с идентичной информацией.

Ключевые слова: шифрование, лавинный эффект, Колмогоровская алгоритмическая метрика.

1. Introduction

There are many papers on satisfactory encryption of single texts. But we have not found additional demands on encryption of sets of texts. Since information protection includes protection of involved persons, we consider this matter.

If the code-breaker intercepts two either identical or similar or having something in common messages then it would be important information about the sender, his methods and the addressees.

We propose: each two ciphertexts (including two encryptions of same plaintext) are to differ sufficiently. To formalize this demand we propose to use asymptotical minimal volume of algorithm of revertible transformation of one text into another one with same information.

This proposal was delivered briefly at the conference [1].

All examples below are illustrative.

2. Kolmogorov complexity and Kolmogorov algorithmic metrics

In algorithmic information theory, the Kolmogorov complexity [2] of an object, such as a piece of text, is the length of a shortest computer program (in a predetermined programming language) that produces the object as output.

Denote length of a word (a text) W as $L(W)$.

We propose

Definition 1. Let T be a set of "sufficiently large" texts with same information. Define the function **(**Kolmogorov algorithmic metrics)

$$\rho_K(t_1, t_2): T \times T \to N$$

as the sum of lengths of shortest computer programs (in a predetermined programming language) transforming a text t_1 to a text t_2 and vice versa (so-called "codec") for $t_1 \neq t_2$; $\rho_K(t_1, t_1)=0$.

Remark. There is "Kolmogorov distance" between distributions. It is not related to Definition 1.

If $\rho_K(t_1, t_2)$ does not depend on lengths of elements of T then such computer programs (metrics) are called "bounded" otherwise they it is called "unbounded".

Example 1 of bounded Kolmogorov algorithmic metrics.

The set T is defined as follows. There is a plaintext t_0 of decimal digits with information. Divide the text t_0 by "fives" and choose a random digit for each "five". Put these digits s_k before each "five" and make annular shift by $(s_k \bmod 5)$ signs in this "five".

We obtain an element (ciphertext) of T.

For instance,

t_0 = '2346798705':

$s_1=6, s_2=0$; t_1 = '672346098705';

$s_1=3, s_2=5$; t_2 = '3467235 98705'.

The codec is obvious.

3. Known and our demands

In cryptography, the avalanche effect is the following: if an input (plaintext) in an encryption algorithm is changed slightly then the output (ciphertext) changes significantly [3]. It is commented as follows:

"If a block cipher or cryptographic hash function does not exhibit the avalanche effect to a significant degree, then it has poor randomization, and thus a cryptanalyst can make predictions about the input, being given only the output. This may be sufficient to partially or completely break the algorithm. Thus, the avalanche effect is a desirable condition from the point of view of the designer of the cryptographic algorithm or device."

We consider it insufficient.

We propose the following demands.

(D1): Each two ciphertexts generated by an encryption algorithm must be sufficiently different. For this purpose the encryption algorithm must involve random numbers which influence all parts of a ciphertext.

(D1'): "sufficiently different" can be formalized by the Kolmogorov algorithmic metrics:

$$\rho_K(t_1, t_2) > 2/3*min\{L(t_1), L(t_2)\}.$$

The demand (D1) can be specified also as

(D1'') "none (lengthy) sub-words of each two ciphertexts are similar" and can be formalized, for instance, as for

$$\rho_K*(u_1, u_2) := min\{\rho_K(w_1, w_2)\} :$$

$$w_1 \subset u_1, w_2 \subset u_2, L(w_1) > L(u_1)/2, L(w_2) > L(u_2)/2\}:$$

$$\rho_K*(u_1, u_2) \geq min\{L(u_1), L(u_2)\}/2.$$

Example 2. Consider the encryption algorithm in Example 1 (for sufficiently large plaintexts): choose s_k randomly. Then each two ciphertexts would be different but not sufficiently different (the text of such algorithm is short).

(D2): a ciphertext must be sufficiently long (otherwise the code-breaker guesses that the plaintext was short).

(D3): a ciphertext must not contain any evident or ciphered information about the sender and the addressee but must contain ciphered information on authenticity.

The addressee is to check all obtained messages for such authenticity.

Example 3. Additional algorithm: after applying the encryption algorithm in Example 2 find the sum of all digits (mod 100) and insert the first digit in the seventh place and do the second digit in the fourth place.

The reverse algorithm is obvious.

We have for t_2: the sum is $3+4+6+7+2+3+5+9+8+7+0+5=59$.

The result is: t_3 = *'34697235598705'*.

Remark. It is very difficult for a code-breaker to hack such method of authenticity because in the best case scenario he could be able to intercept only dozens of suspected messages.

The strict avalanche criterion is a formalization of the avalanche effect. It is the following: whenever a single input bit is complemented, each of the output bits changes with a 50% probability [4].

The demand (D1) overlaps this criterion. Even without complementation, by involving random numbers after applying the same encryption algorithm each of the output bits changes with a 50% probability.

4. Conclusion

The novelty of our proposal can be substituted as follows. If such demands were in use earlier then the avalanche effect was not mentioned in literature at all. We hope that our demands are easy to be implemented and would rise level of information protection.

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MSC 91F20

CLASSIFICATION OF VERBS BY THEIR MATHEMATICAL MODELS

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Supra, the authors proposed independent computer presentation of natural languages and implemented some notions of Kyrgyz and English. In the paper new classification of verbs by their mathematical and computer models is proposed. It can be used for further development of such computer presentations.
Keywords: classification, verb, computer presentation, mathematical model, independent presentation.

Мурда авторлор тарабынан табигый тилдерди компьютерде көз карандысыз түрдө чагылдыруу сунушталган жана кээ бир түшүнүктөр кыргыз жана англис тилдеринде ишке ашырылган. Бул макалада этиштердин математикалык жана компьютердик моделдеринин боюнча жаңы классификациялоосу сунушталат. Бул ушундай компьютердик чагылдырууну андан ары өркүндөтүү үчүн колдонулушу мүмкүн.
Урунттуу сөздөр: классификациялоо, этиш, компьютердик чагылдыруу, математикалык модель, көз карандысыз чагылдыруу.

Ранее авторы предложили независимое компьютерное представление естественных языков и реализовали некоторые понятия кыргызского и английского языков. В статье предлагается новая классификация глаголов по их математическим и компьютерным моделям. Это может быть использовано для дальнейшего развития таких компьютерных представлений.
Ключевые слова: классификация, глагол, компьютерное представление, математическая модель, независимое представление.

1. Introduction

We proposed to develop interactive computer presentations of natural languages. If a computer presentation does not depend on the user's knowledge and skills on similar objects then we call it independent. In our opinion, such presentations are more effective because the user can learn a language inductively, and they begin thinking in it, without translation in mind.

Earlier, investigating and learning a living language were implemented with the assistance (including bilingual dictionaries and text-books) of persons who had a complete command of it. Invention of recording sounds gave possibility to fix examples of an oral language objectively. Invention of talking pictures fixed examples of phrases with connection to situations and actions. Computer games gave the user the opportunity to choose actions with corresponding phrases. Existing software to learn languages base on languages native to the user, nevertheless some notions are presented independently. This survey demonstrates that there were not completely independent presentations of natural languages.

Using ideas [1], [2], [3] we [4-11] gave definitions and developed elements of such presentations. We proposed to use random generation of tasks and situations and feedback for checking-up knowledge of a language. We described mathematical models in general [12].

For further developing of such presentations we need a corresponding classification of notions (nouns and verbs) of languages. Existing classifications are based on involved grammar forms only and are scanty for computer presentation.

We will base on Kyrgyz language mentioning other languages too.

2. Definitions for independent presentation of notions

Definition 1. If low energetic outer influences can cause sufficiently various reactions and changing of the inner state of the object (by means of inner energy of the object or of outer energy entering into object besides of such influences) at any time then such (permanently unstable) object is an *affectable object*, or a *subject*, and such outer influences are *commands*.

Definition 2. A system of commands such that any subject can achieve desired efficiently various consequences from other one is a *language*.

Hypothesis 1. A human's genuine understanding of a text in a natural language can be clarified by means of observing the human's actions in real life situations corresponding to the text.

Definition 3. *Simple* mathematical models consist of fixed (F_i) and movable (M_j) sets and temporal sequence of conditions of types $(M_j \subset F_i)$, $(M_j \cap F_i = \varnothing)$, $(M_j \cap F_i \neq \varnothing)$.

Remark. *Animated* objects and Avatar are denoted by trembling.

Kinematical mathematical models also include *self-moving* objects.

For example, “Where WILL be the [slow-moving] turtle?”

Multi-media mathematical models also include sounds etc.

For example, “Give the nut to the SINGING bird!”

Transformation mathematical models also include *transforming* objects (tools) and *transformable* objects.

For example, “CUT the needle WITH the knife!”

Computer interactive presentations are built on the base of mathematical models.

Definition 4. Let any *notion* (word of a language) be given. If an algorithm acting at a computer: generates (randomly) a sufficiently large amount of instances covering all essential aspects of the *notion* to the user, gives a command involving this notion in each situation, perceives the user's actions and performs their results clearly on a display, detects whether a result fits the command, then such algorithm is said to be a *computer interactive presentation* of the *notion.*

Certainly, commands are to contain other words too. But these words must not give any definitions or explanations of the notion.

Definition 5. If all words being used in Definition 4 are unknown to the user nevertheless s/he is be able to fulfill the meant action (because it is the only natural one in this situation) then the notion (word of a language) is said to be *primary*. If the user has to know supplementary words to complete the action then the notion is said to be *secondary*. Thus, there arises a natural hierarchy of notions.

3. Survey of known classifications of verbs

Such classifications are based on grammar forms only.

3.1. Transitive and intransitive verbs

Transitive verbs in Kyrgyz language are detected by using *Табыш жөндөмө* (Accusative Case).

Remark. Many of intransitive verbs can be made transitive by means of affixes of *Кат буйрук* (Causative voice), for instance ӨЗГӨРҮҮ (transform yourself) - ӨЗГӨРТҮҮ (transform anything).

3.2. Valency is the number and type of arguments controlled by a predicate, content verbs being typical predicates.

There are several types of valency (in Kyrgyz language):

intransitive (monovalent/monadic) УКТА! (Sleep!);

transitive (divalent/dyadic) ТОПТУ ЫРГЫТ! (Throw the ball!);

ditransitive (trivalent/triadic) КЫЗГА ТОПТУ ЫРГЫТ! (Throw the ball to the girl!);

tritransitive (quadrivalent/quadradic) КОЛДОРУҢ МЕНЕН КЫЗГА ТОПТУ ЫРГЫТ! (Throw the ball to the girl with your hands!).

The minimum for the verb ЫРГЫТУУ (throw) is “divalent”.

Remark. On the base of this definition we put

Hypothesis 3. Any notion has a minimalistic mathematical model (involving minimal number of *entities* in Occam’s sense).

These definitions are scanty for computer presentation.

3. Mathematical models for verbs

3.1. Intransitive verbs

3.1.1. “Inner verbs” - due to Definition 1, their meanings are “Change your inner state”.

Examples: 1) ОЙЛОО - think; 2) УКТОО - sleep; 3) (МАСЕЛЕНИ) ЧЫГАРУУ - solve (a task); 4) УГУУ - hear, listen.

They are difficult to build evident mathematical models even with multimedia.

3.1.2. Avatar verbs

“Self-transitive verbs” can be explained as “Change your outer state”. Meanings of many transitive verbs can be returned to the doer, for instance “paint (anything)”- “paint yourself”. Some actual intransitive verbs can be presented in such a way. In mathematical models an (auxiliary, random) thing standing for direct object is to be changed to Avatar.

Examples: 5) (move yourself) БАРУУ - go; 6) КИРҮҮ - enter; 7) ЧЫГУУ - exit; 8) СҮЗҮҮ - swim.

3.2. Transitive verbs

3.2.1. Non-influencing verbs

Examples: 9) ОКУУ - read; 10) КӨРҮҮ - see; 11) КАРОО - look;

12) ТАБУУ - find [but do not touch].

3.2.2. Influencing verbs

Examples: 13) КОЮУ - put; 14) АЛУУ - take; 15) ЖЫЛДЫРУУ - move.

3.2.3. Effecting verbs

Such verbs are related to affectable objects only.

Examples: 16) БЕРҮҮ - give; 17) ТАМАКТАНДЫРУУ - feed.

Example 18 of influencing-effecting trivalent verb: КӨРСӨТҮҮ - show (anything to anybody).

3.2.4. Transforming and tool verbs.

Examples: 19) БҮКТӨӨ - flex; 20) КЕСҮҮ - cut.

4. Conclusion

This paper is a next contribution to our general project of developing mathematical models of various notions for independent presentation of natural languages. We hope that such software would be interesting and useful for people to learn languages.

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MSC 18C05

CATEGORY OF IRGÖÖ-TYPE PROCESSES IN COMPUTATIONAL MATHEMATICS AND ALGORITHMS TO DETECT PATTERNS

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Processes of generation of order from chaos are considered. The authors propose to call them irgöö-type processes by the name of the first process of local diminishing of entropy due to dissolving of energy mentioned in literature. The authors distinguish main features of such processes: they are real (including running computers) and random, are defined by some components (states of computer) at each moment. Supra, by the authors' definition, appearance of phenomena in systems only with large number of components is said to be the effect of numerosity. The least number of components preserving such phenomenon was said to be the constant related to it. Algorithms to detect arising patterns are proposed also.
Keywords: order, chaos, irgöö, numerosity, effect, phenomenon, differential equation, difference equation.

Башаламандыктан тартип пайда болгону процесстери каралат. Авторлор бул процесстерди, адабиятта биринчи белгилүү болгон энергия тарткатылганынан энтропиянын жергилик азаюу процессинин аталышы боюнча алганда, «иргөө» тибиндеги процесстер деп атоону сунуш кылышат. Авторлор мындай процесстердин негизги өзгөчөлүктөрүн төмөнкүчө белгилешет: алар чыныгы (анын ичинде компьютерлердин аракеттери) жана кокустан, ар бир учурда бир нече компоненттер (компьютердин абалдары) менен аныкталат. Мурда, авторлор сунуштаган аныктама боюнча, көп элементтерден турган системалар гана үчүн пайда болуучу кубулуштар «көпчө» эффектиси деп аталган. Мындай кубулушка алып келүүчү, ошол кубулуш менен байланышкан, эң кичине сан турактуу болуп саналат. Ошондой эле пайда болгон закон ченемдүүлүктөрдү аныктоо үчүн алгоритм сунушталган.

Урунттуу сөздөр: ирет, башаламандык, иргөө, «көпчө», эффект, кубулуш, дифференциал-дык теңдеме, айырмалуу теңдеме.

Рассматриваются процессы возникновения порядка из хаоса. Авторы предлагают называть их процессами типа «иргөө» по названию первого такого процесса локального уменьшения энтропии вследствие диссипации энергии, известного в литературе. Авторы выделяют основные признаки таких процессов: они - реальные (включая действия компьютеров) и случайны, определяются несколькими компонентами (состояниями компьютера) в каждый момент. Ранее, по определению авторов, возникновение явлений только для систем с большим количеством компонент названо эффектом «множественности». Самое малое число, вызывающее такое явление, названо постоянной, связанной с этим явлением. Также разработан алгоритм для определения возникающих закономерностей.

Ключевые слова: порядок, хаос, иргөө, множественность, эффект, явление, дифференциальное уравнение, разностное уравнение.

1. Introduction

Processes of generation of order from chaos are considered in the paper. We propose to call them irgöö-type processes by the name of the first such process of local diminishing of entropy due to dissolving of energy mentioned in literature [1]. We distinguish main features of such processes: they are real (including running computers), random, are defined by some components (states of computer) at each moment.

Remark 1. Considering a computer as a real object and computer presentations as real processes was noted in [2].

Remark 2. Our as well as other authors' "Definitions" below are not strongly mathematical because they mean real objects and processes.

Remark 3. Discoveries of new "phenomena" and "effects" used to be sufficient steps in developing science but there were not definitions of these notions before our publication [3]. We gave corresponding definitions and examples, proposed methodic to search new "phenomena" as consequences of "effects".

Remark 4. Differential equations (excluding some simplest ones) are not "processes" in our sense because differential equations in general cannot be "solved" to detect any "phenomena". Because of instability difference equations approximating differential ones can have other properties.

Remark 5. Difference equations (excluding some simplest ones) are not "processes" in our sense because they in general cannot be "solved" to detect any "phenomena". Because of computational errors and instability numerical experiments with them can give other results.

Hence, a program run on a concrete computer only is a "process".

In this paper we consider consequences of the irgöö, or "self-ordering" effect.

Section 2 contains necessary definitions.

Section 3 presents definition of effect of numerosity.

There are examples of some known processes in Section 4 and processes presented on computers in Section 5.

In Section 6 we propose an algorithm to detect a pattern.

2. Definitions

We proposed [4] (improved)

Definition 1. Consider a set of "states". States are "homogenous", uniformly bounded and uniformly bounded from below in any sense.

I1) There is possibility of passing from each state to each other state.

A "possible" process is a (discrete or continuous) sequence of states meets the following conditions.

I2) The process is random and is caused by incoming "purposeless, homogeneous" energy and amount of leaving energy (heat) is approximately equal to one of incoming energy.

Hypothesis 1. If I3) the states have "infinite" components or "sufficiently many" components
and I4) entropy of incoming energy is sufficiently large than one of leaving energy then entropy of states decreases while the process.

It is known that decreasing of entropy is equivalent to arising of ordering.

If such ordering is detected in any way then the process is said to be an irgöö-type process.

There are known

Definition 2 [5]. A dissipative system is a thermodynamically open system which is operating out of, and often far from, thermodynamic equilibrium in an environment with which it exchanges energy and matter.

Definition 3 [6]. Synergetics is an interdisciplinary science explaining the formation and self-organization of patterns and structures in open systems far from thermodynamic equilibrium.

Our Definition 1 is more general. Firstly, items I1), I2) and I3) are not announced but are meant in Definitions 2 and 3.

Secondly, Definitions 2 and 3 mention “thermodynamic equilibrium”, i.e. they include states composed of microparticles (moleculas and atoms) only and do not take in account states composed of macroparticles such as irgöö (see 4.1 below).

3. Effect and constants of numerosity

The law of large numbers can be considered as some phenomena in statistic.

Supra, by our definition, appearance of phenomena in systems only with large number of components was said to be the effect of numerosity.

We found some phenomena due to this effect not related to statistic.

We proposed a definition of constants related to this effect and estimated two of them [7].

Definition 4. Appearance of phenomena in systems only with large number of components is said to be the effect of *numerosity*.

Definition 5. If a phenomenon occurs less often for number of components less than N and does more often for number of components greater than N then the number N is said to be the *constant of numerosity* for this phenomenon.

4. Examples of some irgöö-type processes with infinite states

4.1. Irgöö-type process. States are various arrangements of steady balls of different sizes of same material in a wide symmetrical vessel.

The is random vibration of the vessel.

The phenomenon is "migration of the biggest one to the center of their surface."

Remark 6. This experimental fact is too difficult to be proven by any mathematical model but can be corroborated by numerical experiments, see 5.1 below.

4.2. Rayleigh-Bénard convection. States: planar horizontal layer of fluid with various temperature and motion in various points in a (large flat) vessel.

The process is changes in fluid while the vessel is heated from below.

The phenomenon is formation of approximate regular right hexagonal prisms (Bénard cells).

4.3. Belousov–Zhabotinsky reaction. States: liquid mixes of certain chemicals.

The process is a sequence of chemical reactions.

The phenomenon is non-periodical random changing of two colors of liquid.

4.4. Phenomenon of shaker segregation of loose materials in vibrating vessels [8].

5. Examples of irgöö-type processes on computer

5.1. Computer presentation of a process similar to irgöö process [1].

The program was built as follows.

The cylinder of radius 1 was taken as a vessel. Let a (large) natural number n and (small) positive radiuses $r_1 > r_2 \geq \ldots \geq r_n$ be given.

If a set of n points $\{(x_k, y_k, z_k): k=1..n\} \subset R^3$ fulfills the conditions

1) $r_k \leq z_k$, $x_k^2 + y_k^2 \leq (1 - r_k)^2$ for all k in $1..n$ (all balls are in the vessel)*;*

2) $(x_k - x_j)^2 + (y_k - y_j)^2 + (z_k - z_j)^2 \geq (r_j + r_k)^2$ for all $k \neq j$ in $1..n$ (the balls do not overlap) then such set is said to be *admissible.*

A (short) vector $\{u, v, w\}$ $(w<0)$ is said to be *admissible* for a given admissible set of points and a number k in $1..n$ if the set obtained by means of changing the k-th point to the point (x_k+u, y_k+v, z_k+w) is admissible too. Such passing from one set of points to a new set of points is said to be an *admissible* shift.

While it is possible, in the obtained admissible set of points execute random admissible shifts.

Foe instance, we chose $n=50$ and $r_k=0.3-0.01k$, $k=1..19$; $r_k=0.1$, $k=20..50$.

Some runs of this program confirmed Hypothesis one and gave the constant of numerosity $N \sim 50$.

5.2. A constant related to self-ordering of electrical charges

We searched self-ordering of discrete electrical charges in viscous media [3]. Motion of equal, repelling by the Coulomb law electrical charges from a random initial distribution on a topological torus formed a final regular grid was modeled by computer.

Motion of *N* electrical charges can be described by a system of *N* two-dimensional differential equations. These differential equations were approximated by a system of difference equations.

The following program with graphical demonstration of the initial distribution and of the final one was written in *pascal* (with *N=256*).

```
program sabina; uses crt, graph;
var hxy,vx,vy,dx,dy,dxy,dxy1,hxy1,z,z2,xj,yj,dxy2,dxyd: double;
i,j,nxy,it,nt,np,ihand,n_time,ik: longint;
var drv, mode,f,n: integer; x,y:array[1..300] of double;
xn,yn:array[1..300] of integer;
begin {main} drv:=0; mode:=VgaHi; InitGraph(drv,mode,'c:\tp\bgi');
randomize; SetTextStyle(0,0,2);
OutTextXY(30,20,'Repelling 256 electrical charges on torus');
OutTextXY(100,40,'(Wait a little)'); z:=700.; z2:=z/2.0;
np:=10; {nt:=500*n_time;} hxy:=1.0; hxy1:=hxy; nt:=1000; nxy:=256;
for ik:=1 to nxy do begin  x[ik]:=z*random; y[ik]:=z*random;
xn[ik]:=round(x[ik]); yn[ik]:=round(y[ik]);
Setcolor(green); circle(xn[ik]+80,yn[ik]+70,2); end;
for it:=0 to nt do begin {it} if it>np then hxy:=2.0*hxy1; if it>2*np then
hxy:=4.0*hxy1;
    for i:=1 to nxy do begin {i=ix} vx:=0.; vy:=0.; for j:=1 to nxy do
begin if j<>i then begin
```

```
xj:=x[j]; if xj>x[i]+z2 then xj:=xj-z; if xj<x[i]-z2 then xj:=xj+z;
yj:=y[j]; if yj>y[i]+z2 then yj:=yj-z; if yj<y[i]-z2 then yj:=yj+z;
dxy2:=sqr(x[i]-xj)+sqr(y[i]-yj)+1.;
dxy1:=z/(dxy2*sqrt(dxy2)); if dxy1<sqr(z)/nxy*0.5 then begin
dx:=(x[i]-xj)*dxy1; dy:=(y[i]-yj)*dxy1; vx:=vx+dx; vy:=vy+dy; end; end; end;
 x[i]:=x[i]+vx*hxy; if x[i]>z then x[i]:=x[i]-z; if x[i]<0. then x[i]:=x[i]+z;
 y[i]:=y[i]+vy*hxy; if y[i]>z then y[i]:=y[i]-z; if y[i]<0. then y[i]:=y[i]+z; end {i=ix};
for ik:=1 to nxy do begin xn[ik]:=round(x[ik]); yn[ik]:=round(y[ik]) end; end {it};
Setcolor(white);
repeat for ik:=1 to nxy do begin circle(xn[ik]+80,yn[ik]+70,8);
circle(xn[ik]+80,yn[ik]+70,6); circle(xn[ik]+80,yn[ik]+70,4);
circle(xn[ik]+80,yn[ik]+70,2) end;
delay(100); until keypressed; end.
```

Runs of this program found the constant of numerosity $N \sim 110$.

Also, when the number of charges is a square of even number then the grid is square in most of experiments; when it is a square of odd number then the grid is triangular in most of experiments.

6. An algorithm to detect patterns

For detecting patterns in Example 5.1 we propose

ALGORITHM 1. Let a finite set of K points $\{z[1]..\ z[K]\}$ in a locally being R^n space be given.

A) Found the minimum $M:=\min\{|z[i]-z[j]|:1\le i<j\le K\}$.

B) Choose a constant $M_+ > M$ and close to M and calculate numbers of points

$N_i:=card\{i:M\le |z[i]-z[j]|\le M_+\},\ i=1..k$.

C) If most of numbers N_i are equal (let their common value be N) then a pattern exists.

For example, in R^2: *if* $N=3$ then a hexagonal grid exists;

if $N=4$ then a hexagonal grid exists;

if $N=6$ then a triangular grid exists.

Conclusion

We hope that proposed definitions would yield new phenomena in reality and in computational experiments and constants of numerosity would be found for other real and virtual processes.

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MSC 54D05

SEARCH IN MULTIDIMENSIONAL SPACES AND GENERALIZATION OF TIKHONOV'S THEOREM

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Considered the task on extracting information from evident presentations of unknown objects in kinematical topological spaces on computer. For this purpose well-known Tikhonov's theorem on continuity of reverse function for bijection with compact domain is generalized to surjections. Some examples of applications of this theorem are given.

Key words: topological space, multidimensional space, kinematical space, continuous function, motion, Tikhonov's theorem.

Макалада компьютерде кинематикалык топологиялык мейкиндиктерде белгисиз объекттердин визуалдык көрүнүштөрүнөн керектүү маалыматтарды алуу маселеси каралат. Ошол максатта аныктоо аймагы компакттык болгон биекция үчүн Тихоновдун тескери функциясынын үзгүлтүксүздүгү жөнүндөгү белгилүү теоремасы сюръекцияга жалпыланган. Бул теореманы колдонууга мисалдар келтирилген.

Урунттуу сөздөр: топологиялык мейкиндик, көп өлчөмдүү мейкиндик, кинематикалык мейкиндик, үзгүлтүксүз функция, кыймылдоо, Тихонов теоремасы.

В статье рассматривается задача извлечения информации из наглядных представлений неизвестных объектов в кинематических топологических пространствах на компьютере. С этой целью известная теорема Тихонова о непрерывности обратной функции для биекции с компактной областью определения обобщена на сюръекции. Приведены примеры применения этой теоремы.

Ключевые слова: топологическое пространство, многомерное пространство, кинематическое пространство, непрерывная функция, движение, теорема Тихонова.

1. Introduction

The paper deals with the general problem to obtain any information about an object x by results y of remote "observing" it (certainly, y depends on x continuously).

The main result on possible correctness of such problem was obtained by Tikhonov: if the space of possible objects x is compact and different x's generate different y's then the reverse function restoring x by y is continuous.

Meanwhile sometimes we need not the whole object x but any information on it. This paper deals with it.

The second section presents the theorem on extraction of information.

Definitions for motion in virtual spaces are given in the third section.

The fourth section contains examples on application of this theorem.

This result was published briefly [6].

2. Theorem on extracting information

We will consider metrical spaces only.

Tikhonov's

Theorem 1 [1]. If X is a compact space and $f : X \to Y$ is a bijection and continuous then the inverse function $f^{-1} : Y \to X$ is continuous.

This theorem is used to prove correctness of inverse problems where X contains "hidden" objects, Y does observable ones and f is a kind of projection or integration.

But sometimes f is not an injection. One the other hand, not elements themselves of X but some their features are needed.

Let these features be expressed in a continuous function $P: X \to Z$ where Z is a space of "features".

Theorem2. If

1) X is a compact space;

2) $f : X \to Y$ is a surjection and continuous;

3) $(f(x_1) = f(x_2)) \Rightarrow (P(x_1) = P(x_2))$ then the assertion $(\exists x \in X)((y = f(x)) \wedge (z = P(x))$ defines a continuous function $g:Y \to Z.$

Proof. Let $y_0 \in Y$. Due to 2) $(\exists x_0 \in X)(y_0 = f(x_0))$ and $(\exists z_0 \in Z)(z_0 = P(x_0))$. Hence, the function g is defined.

Further, if $((y_0 = f(x_1)) \wedge (y_0 = f(x_2))$ then $P(x_1) = P(x_2)$ by 3).

Hence, the function g is defined uniquely.

Suppose that (*) there exists such sequence $\{x_k : k = 1,2,3,...\} \subset X$ that the sequence

$$\{y_k := f(x_k): k = 1,2,3,..\}$$

converges to y_0 but the sequence

$$\{z_k := P(x_k) : k = 1,2,3,..\}$$

does not converge to z_0.

Then there exists such $\varepsilon > 0$ that any infinite subset Z_1 of the set $\{z_k : k = 1, 2, 3, ...\}$ is out of E_0 := (ε-neighborhood of z_0).

Consider the corresponding subset X_1 of the set $\{x_k: k = 1, 2, 3, ...\}$.

By 1) there exists a subset $X_2 \subset X_1$ converging to any $x' \in X$. Then the corresponding subset $Z_2 \subset Z_1$ converges to $z' := P(x')$ being out of E_0.

Hence, $P(x') \neq P(x_0)$. But $f(x') = f(x_0)$.

Therefore, the assumption (*) has implied a contradiction. Consequently, the function g is continuous. Theorem is proven.

We proposed [2], [3] motion of lengthy objects in kinematical spaces. Different positions of same objects can give equal projections. But if we are interested in detecting other features of these objects then Theorem 2 yields possibility for it by projections.

3. Survey of axiomatization of motion in spaces with bounded velocity

As a codifying the ancient idea of controlled motion with bounded velocity, the notion of kinematical space was introduced [2]. Briefly,

Definition 1. A computer program is said to be a **presentation** of a computer kinematical space if:

P1) there is an (infinite) metrical space X of points and a set X_1 of display-presentable points being sufficiently dense in X;

P2) the user can pass from any point x_1 in X_1 to any other point x_2 by a sequence of adjacent points in X_1 by their will;

P3) the minimal time to reach x_2 from x_1 is (approximately) equal of the minimal time to reach x_2 from x_1.

The space X is said to be a **kinematic space**; the space X_1 is said to be a **computer kinematic space**; this minimal time is said to be the **kinematical distance** ρ_X between x_1 and x_2; a sequence of adjacent points is said to be a **route**.

If there exists a kinematic consistent with the given metric then the metric space is said to be **kinematizable**.

We proposed more general definition [3], [4], [5]. We considered the practical task. Let there be a thing and obstacles. It is necessary to move the thing to another place. Is possible? If yes then in what minimal time it can be done?

Definition 2. There is a family P of connected subsets of the set X (we will call them **passes**); each pass has the positive **length (time)** and a family Q of connected subsets of the set X (we will call them **things**).

(It means that a thing moves along a pass).

The space X is said to be a **generalized kinematic space**.

(G1) For each $x \in p \in P$ there exists such $q \in Q$ that $x \in q$ [a thing can be in each place of a pass].

(G2) For each $x_1 \neq x_2 \in X$ there exists such pass $p \in P$ that $x_1, x_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between points x_1 and x_2.

(G3) For each $q_1 \neq q_2 \in Q$ there exists such pass $p \in P$ that $q_1, q_2 \in P$ and the set of lengths of such p is bounded with a positive number below; this infimum is said to be the **generalized kinematical distance** ρ_X between things p_1 and p_2.

(G4) If $x_1, x_2 \in p_1$ and $x_2, x_3 \in p_2$ then there exists such pass $p_3 \in P$ that $x_1, x_2, x_3 \in p_3$ and $length(p_3) \leq length(p_1) + length(p_2)$.

If (G5) For each $x_1 \neq x_2 \in X$ there exists such pass $p_{12} \in P$ that $length(p_{12}) = \rho_X (x_1, x_2)$ then the generalized kinematical space X is said to be **flat** (with respect to P).

If there exists a generalized kinematic (families P and Q) consistent with the given Hausdoff metric for Q then the metric space is said to be K-kinematizable.

4. Examples of extracting information

These examples are illustrative.

Example 1. Let X be the set of segments on a plane with Hausdorff metric, $Y := R_+^2$, $Z := R_+$; *f(x):={length of projection of x on the abscissa, length of projection* *of x on the ordinate}; P(x)* is the length of *x*.

Then, by Theorem 2, a continuous function $g\colon Y \to Z$ exists. Hence, the length of a segment can be found by its projections although the segment itself cannot be found.

Example 2. Let X be the set of segments in R_+^n with Hausdorff metric, $Y:=R_+^n$, $Z:= R_+$;

f(x):={length of projection of x on the x_1*-axis, length of projection of x on the* x_2*-axis, …, length of projection of x on the* x_n*-axis};*

P(x) is the length of *x*.

Then, by Theorem 2, a continuous function $g\colon Y \to Z$ exists. Hence, the length of a segment in R_+^n can be found by its projections although the segment itself cannot be found.

Example 3. Let X be the set of linear functions of $C[-1,1] \to R$, $Y:=R$, $Z:= R_+$;

$f(x):=\int_{-1}^{1} x(t)tdt$; *P(x)* is the slope *b* of the function $x(t)=a+bt$.

We have: $\int_{-1}^{1}(a+bt)tdt = 2b/3$.

Then, by Theorem 2, a continuous function $g\colon Y \to Z$ exists. Hence, the slope of a linear function can be found by an integral of it although the function itself cannot be found.

5. Conclusion

We hope that the new definitions in this paper would provide more effective ways of investigation of unknown objects in virtual and real spaces.

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MSC 45D05

ALGORITHM TO INVESTIGATE LINEAR VOLTERRA INTEGRAL EQUATIONS WITH PROPORTIONAL RETARDING OF ARGUMENT

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Supra the author constructed and implemented the following algorithms on a computer. Given an equation of Volterra type with power coefficients by integral summands, the algorithm presents data to detect existence of solutions and occurrence of arbitrary constants in it; also, given an equation with a coefficient, specific values of the coefficient are found. In this paper such items are considered for integral equations with proportional retarding of argument.
Keywords: integral equation, unbounded domain, Volterra equation, algorithm, analytical function.

Мурда, автор төмөнкү алгоритмдерди түзүп жана компьютерде жүзөгө ашырган. Даражалуу көбөйтүндүлүү интегралдык кошулуучулары бар теңдеме берилген. Теңдемеде үчүн алгоритм чыгарылышынын жашоосун аныктоо жолуна мүмкүндүгүн жана анда каалагандай турактуу сан бар экендигин аныктоо үчүн малыматты берет. Ошондой эле коэффициент менен теңдеме сунушталган, бул теңдемеде коэффициенттин өзгөчө маанисин

табуу, анализдөө маселеси каралат. Бул макалада бул сыяктуу маселелер аргументинин кечигүүсү пропорциялуу болгон интегралдык теңдемелер үчүн каралат.
Урунттуу сөздөр: интегралдык теңдеме, чектелбеген аймак, Вольтерра тибиндеги теңдеме, алгоритм, аналитикалык функция.

Ранее автор построила и реализовала на компьютере следующие алгоритмы. Дано уравнение со степенными сомножителями при интегральных слагаемых, алгоритм представляет данные для определения существования решения и наличия в нем произвольных постоянных; также дано уравнение с коэффициентом, находятся особые значения коэффициента. В данной статье такие вопросы рассматриваются для интегральных уравнений с запаздывающим аргументом.
Ключевые слова: интегральное уравнение, неограниченная область, уравнение типа Вольтерра, алгоритм, аналитическая функция.

Introduction

Before our publications, we did not find algorithms on conditions of existence of solutions of Volterra equations with analytical functions. Supra we constructed and implemented the following algorithms on a computer [1-4]. Given a linear equation with power coefficients by integral summands, the algorithm presents data to detect existence of solutions and occurrence of arbitrary constants in it; also, given an equation with a coefficient, specific values of the coefficient are found.

In this paper such items are considered for integral equations with proportional retarding of argument.

1. Statement of problem

We will use denotations

$$\boldsymbol{R} := (-\infty;\infty);\ \boldsymbol{R}_{+} := [0;\infty);\ \boldsymbol{R}_{++} := (0;\infty);\ \boldsymbol{N}_0 := \{0, 1, 2, 3, ...\};\ \boldsymbol{N} := \{1, 2, 3, ...\}.$$

We will use the term ”Algorithm” as it is usually understood in Analysis: arithmetical operations and comparison over numbers in $\boldsymbol{R}$ (for rational numbers this definition coincides with the strict one).

We will write discrete arguments in brackets to bring denotations nearer to algorithmic ones and to bypass the common ambiguity of expressions such as a_{2j}.

We will consider equations of type

$$tu(t) + \sum_{a,p} bt^p \int_0^{at} u(s)ds = f(t), \qquad (1)$$

$0 < a \le 1, p \ge 0$, $f(t)$ is an analytic function, $f(0) = 0$.

In other words, we consider equations with terms:

Integral operators $I_\alpha u(t)$, $I_\alpha u := \int_0^{at} u(v)dv$ with coefficients bt^p.

There are not two terms with equal p and a in (1).

We will consider given and unknown real-valued analytical functions in the form

$$f(t) = f[1]t + f[2]t^2 + \ldots,$$

$$u(t) = u[0] + u[1]t + u[2]t^2 + \cdots.$$

Substituting in (1) we obtain

$$t(u[0] + u[1]t + u[2]t^2) + \cdots + \Sigma bt^p \int_0^{at} (u[0] + u[1]s + u[2]s^2 + \ldots)ds =$$

$$= f[1]t + f[2]t^2 + \cdots \;;$$

$$t(u[0] + u[1]t + u[2]t^2 + \cdots) + \Sigma bt^p(atu[0] + a^2t^2u[1]/2 + a^3t^3u[2]/3 +$$

$$\cdots) = f[1]t + f[2]t^2 + \cdots \;;$$

$$u[0]t + u[1]t^2 + u[2]t^3 + \cdots +$$

$$+\Sigma(abu[0]t^{p+1} + a^2bu[1]/2 \cdot t^{p+2} + a^3bu[1]/3 \cdot t^{p+3} + \cdots) =$$

$$= f[1]t + f[2]t^2 + \cdots . \qquad (2)$$

Gathering coefficients by $t, t^2, t^3, \ldots$ terms we obtain a system of difference equations for $u[0], u[1]$, ... Denote it as (3).

2. Algorithm

We propose the following

Algorithm.

For a human:

1) Input the number (K) of integral operators.

2) Subsequently for $k=1..K$ input:

the non-negative integer number p_k (in increasing order);

the rational number $a_k \in (0,1]$ (in increasing order for equal values of p_k);

the non-zero rational number b_k.

For a computer:

3) Portray the integral equation for the function $u(t) = u[0] + u[1]t + \ldots$.

By custom, the cases $b_k=-1$ and $b_k=1$ are demonstrated individually; the cases $p_k=0$ and $p_k=1$ are demonstrated individually.

4) Estimate the maximal value (m_0) of exponents after which the total coefficients by elder unknowns will be non-zero (see Theorem below);

5) Portray the system (3) for exponents $1 .. m_0+1$ of t.

The result is the system of equations for $u[0], u[1]$, ...

There may be “0” or “0*u[k]” in the left-hand sides of some equations.

For a human:

6) Investigate the system of linear algebraic equations (3). What of $u(0)$, $u(1)$, ... can be found under conditions on *f [0], f [1], ...?*

3. Main theorem

Theorem 1. There exists such value (m_0) of exponents that after it the total coefficients by elder unknowns are be non-zero.

Proof. I case. p[K]=0. (2) is the following:

$$u[0]t + u[1]t^2 + u[2]t^3 + \cdots +$$
$$+\Sigma_{k=L}^{K} (a[k]b[k]u[0]t + a[k]^2 b[k]u[1]/2 \cdot t^2 + \cdots) =$$
$$= f[1]t + f[2]t^2 + \cdots . \quad (4)$$

$$t^m: u[m-1] + \Sigma_{k=L}^{K} a[k]^m b[k]u[m-1]/m = f[m].$$

If $m > m_0 := \Sigma_{k=L}^{K} a[k]|b[k]|$ then $u[m-1]$ can be found.

II case. p[K]=1. (2) is the following:

$$u[0]t + u[1]t^2 + u[2]t^3 + \cdots +$$
$$+\Sigma_{k=Q}^{L-1} (a[k]b[k]u[0]t + a[k]^2 b[k]u[1]/2 \cdot t^2 + \cdots) +$$
$$+\Sigma_{k=L}^{K}(a[k]b[k]u[0]t^2 + a[k]^2 b[k]u[1]/2 \cdot t^3 + \cdots) =$$
$$= f[1]t + f[2]t^2 + \cdots . \quad (5)$$

$t^m (m > 1)$:

$$u[m-1] + \Sigma_{k=Q}^{L-1} a[k]^m b[k] u[m-1]/m +$$
$$+\Sigma_{k=L}^{K} a[k]^{m-1} b[k] u[m-2]/(m-1) = f[m].$$

If $m > m_0 \coloneqq \Sigma_{k=Q}^{L-1} a[k]|b[k]|$ then $u[m-1]$ can be found by means of $u[m-2]$. The general case is similar to this case.

4. An example of integral equation

We consider the equation

$$tu(t) - 12\int_0^{t/2} u(s)ds = f(t). \quad (6)$$

Here

K=1; p[1]=0; a[1]=1/2; b[1]=–12.

Equation

t u(t) – 12 int_0^(1/2) t u(s)ds = f(t).*

Substituting:

$$u[0]t + u[1]t^2 + \ldots - 12\int_0^{t/2} (u[0] + u[1]s + \ldots)ds = f[1]t + f[2]t^2 + \ldots$$

$$u[0]t + u[1]t^2 + \ldots - 12(u[0]t/2 + u[2]t^2/3/4 + \ldots) = f[1]t + f[2]t^2 + ..$$

Hence, a solution exists if $f[2] = 0$.

5. Conclusion

We hope to construct such algorithms for various classes of integral equations.

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MSC 54E15, 54D20

UNIFORM MENGER SPACE

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In this work we study some properties of the uniformly Menger space introduced by Lj. D. Kocinac.
Key words: Uniform space, uniform Menger space, uniformly continuous mapping, uniformly perfect mapping.

Бул илимий макалада Л.Д. Кочинац киргизген бир калыптуу Менгер мейкиндигинин кээ бир касиеттери изилденилет.
Урунттуу сөздөр: Бир калыптуу мейкиндик, бир калыптуу Менгер мейкиндиги, бир калыптуу үзгүлтүксүз чагылдыруу, бир калыптуу жеткилең чагылдыруу.

В работе изучается некоторые свойства равномерно Менгера пространство введенное Л.Д. Кочинацом.
Ключевые слова: Равномерное пространство, равномерное Менгера пространство, равномерно непрерывное отображение, равномерно совершенное отображение.

In this article, we study some properties of a uniformly Menger space in the sense of Lj. D. Kocinac.

Let us recall some of the most important concepts from the theory of selection principles in topological spaces and uniform spaces and uniformly continuous mappings.

Let A and B be collections of open covers of a topological space X. The symbol $S_1(A,B)$ denotes the selection hypothesis that for each sequence $\{U_n : n \in N\}$ of elements of A there exists a sequence $\{V_n : n \in N\}$ such that for each $n, M_n \in U_n$ and $\{M_n : n \in N\} \in B$ [3], [4].

The symbol $S_{tin}(A,B)$ denotes the selection hypothesis that for each sequence $\{U_n : n \in N\}$ of elements of A there is a sequences $\{V_n : n \in N\}$ such that for each $n \in N$, V_n is a finite subset of U_n and $\bigcup_{n \in a_i} V_n$ is an element of B [3], [4].

If O denotes the collection of all open covers of a space X, then X is said to have the Menger property is true for X [3], [4].

Let α be a cover set of X and $L \subset X$ a subset. Let $St(L,\alpha) = \{A \in \alpha : A \cap L \neq \emptyset\}$. For $L = \{x\}, x \in X$, we write $St(x,\alpha)$ instead of $St(\{x\},\alpha)$ [1], [2].

For coverings α and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$. The set $\alpha(x) = \cup St(x,\alpha)$ is called the star of the point x with respect to the cover α, and the set $\alpha(L) = \cup St(L,\alpha)$ is called the star set L with respect to the covering α [1], [2], [5].

For coverings α and β of the set X, the symbol $\alpha \triangleright \beta$ means that the covering α is a star refinement of the covering β, i.e. for any $x \in X$ there exists $B \in \beta$ such that $\alpha(x) \subset B$. For coverings α and β of the set X, the symbol $\alpha * \succ \beta$ means that the covering α is a strongly starrefinement of the covering β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$ [1], [2].

A sequence $\{\alpha_n : n \in N\}$ of coverings of a set X is called normal if $\alpha_{n+1} * \alpha_n$ for any $n \in N$ [1].

Let (X,U) be a uniform space. Each uniformity U on a set X naturally generates some topology τ_U on X. The topology τ_U is called the topology generated or induced by the uniformity U. If an X is a Tychonoff space and a uniformity U on a set X induces the original topology X, then U is said to be a uniformity on a Techonoff space X.

By U_X we denote the universal uniformity of a Tychonoffspace X.

A uniform space (X,U) is called precompact if it has a base consisting of finite coverings. A uniform space (X,U) is called $\aleph_0$-bounded (or pre-Lindelöf), if it has a base consisting of countable coverings. A uniform space (X,U) is called completely bounded if for any $\alpha \in U$ there exists a finite set $M \subset X$ such that $\alpha(M) = X$ [1], [2].

A map $f : X \to Y$ of a topological space X into a topological space Y is called perfect if it is closed and densely compact, then the preimage $f^{-1}y$ of each point of $y \in Y$ is compact. A mapping $f:(X,U) \to (Y,V)$ of a uniform space (X,U) to a uniform space (Y,V) is called precompact if, for any $\alpha \in U$, there exists a finite cover $\gamma \in U$ and a cover $\beta \in V$ such that $f^{-1}\beta \wedge \gamma \succ \alpha$. A map $f:(X,U) \to (Y,V)$ of a uniform space (X,U) into a uniform space (Y,V) is called uniformly perfect if it is precompact and perfect in the topological sense [1].

The uniform space (X,U) is uniform Menger space, if for each sequence $\{\alpha_n : n \in N\} \subset U$ there is a sequence $\{\beta_n : n \in N\}$ such that for each $n \in N$ β_n is a finite subset of α_n and $\bigcup_{n \in N} \beta_n$ is a cover of X, i.e. for each sequence $\{\alpha_n : n \in N\} \subset U$ there is a sequence $\{A_n : n \in N\}$ of finite subsets of X such that $X = \bigcup_{n \in N} St(A_n, \alpha_n)$ [3].

Proposition 1. If a Tychonoff space X possesses the Menger property, then the uniform space (X,U) possesses the uniformly Menger property, i.e.is a uniformly Menger space.

Proof. Suppose that a Tychonoff space X has the Menger property and $\{\alpha_n : n \in N\} \subset U$ is a sequence of uniform coverings. We put $\{\alpha_n : n \in N\}$ where $\langle \alpha_n \rangle$ is the interior of α_n. It is clear that $\{\alpha_n : n \in N\} \subset U$. Since a Tychonoff space X possesses the Menger property, for a sequence of open coverings $\{\alpha_n : n \in N\}$ there exists a sequence $\{\beta_n : n \in N\}$ such that for each $n \in N$, β_n is a finite subfamily of α_n and $\bigcup_{n \in N} \beta_n$ is an open cover in X. Consequently, (X, U) is a uniformly Menger space.

However, if a uniform space (X, U) is a uniformly Menger space, then its topological space does not necessarily have the Menger property. For example, let X be a non-Lindelöf topological space. Then the uniform space (X, U_ρ), where U_ρ is a precompact uniformity, is a uniformly Menger space, but its topological space X is not a Menger space.

Proposition 2. Any precompact space (X, U) is a uniformly Menger space.

Proof. Let (X, U) be a precompact uniform space and $\{\alpha_n : n \in N\} \subset U$ be an arbitrary sequence of uniform coverings.Then for any $n \in N$ there is a finite uniform cover $\beta_n \in U$ such that $\beta_n \succ \alpha_n$. It is easy to see that α_n^o is a finite subcover of the covering α_n for any $n \in N$, that is, $\{\alpha_n^o\}$ is a sequence of finite covers.Also, it is easy to see that $\bigcup_{n \in N} \alpha_n^o$ forms a cover for X. Consequently, (X, U) is a uniformly Menger space.

Proposition 3. Any uniformly Menger space is $\aleph_0$-bounded.

Proof. Let (X, U) be a uniformly Menger space and $\alpha \in U$ an arbitrary uniform cover.Put $\{\alpha_n : n \in N\}$, where $\alpha_n = \alpha$ for any $n \in N$. Then there is a sequence $\{\beta_n\}$ such that β_n is a finite subfamily of α_n and $\bigcup_{n \in N} \beta_n$ is a cover.The cover $\bigcup_{n \in N} \beta_n$ is countable as the union of countable many finite subfamilies. Therefore, X is $\aleph_0$ -bounded.

Proposition 4. A Tychonoff space X is a Menger space if and only if (X,U_X) is a uniformly Menger space, where U_X is a universal uniformity.

Proof. Necessity. Let X be a Menger space. It is laconic that (X,U_X) is a uniform Mengerspace. Let $\{\alpha_n : n\in N\}$ be a sequence of open uniform coverings. Then there is a sequence $\{\beta_n : n\in N\}$ of open families such that for any $n \in N$ β_n there is a finite subfamily for α_n and $\bigcup_{n\in N}\beta_n$ an open cover. Hence, (X,U_X) is a uniformly Menger space.

Sufficiency. Let (X,U_X) be a uniformly Menger space and $\{\alpha_n : n\in N\}$ is a sequential open cover.Since U_X is a universal uniformity, then $\{\alpha_n : n\in N\}\subset U_X$ .Then there is a sequence $\{\beta_n : n\in N\}$ such that for any $n \in N$ β_n is a finite subfamily of α_n and $\bigcup_{n\in N}\beta_n$ is an open cover of X . Therefore, X is a Mengerspace.

Theorem 1. Under uniform perfect mappings, the Menger property is uniformly preserved in both directions.

Proof. The preservation of the properties of the uniformly Menger property towards the image is actually contained in the theorem in (see [3], p. 131). Now let us prove that Menger uniformly preserves the property towards the preimage.Let $f:(X,U)\to(Y,V)$ be a uniformly perfect mapping of a uniform space (X,U) into a uniform Mengerspace (Y, V) and $\{\alpha_n : n\in N\}$ an arbitrary sequence of uniform coverings of the spaces (X,U). Since f is uniformly perfect, for any $\alpha_n\in\{\alpha_n : n\in N\}$ there exists a finite uniform cover $\gamma_n \in U$ and a uniform cover $\beta_n \in V$, $f^{-1}\beta_n \wedge \lambda_n \succ \alpha_n$. Put $\{\beta_n : n\in N\}$. Then for a sequence $\{\beta_n : n\in N\}\subset V$ of uniform covers there is a sequence $\{\delta_n : n\in N\}$ such that for any $n \in N$ δ_n is a finite subfamily of β_n and $\bigcup_{n\in N}\delta_n$ is a cover of (Y,V). Then for any $n \in N$ $f^{-1}\delta_n$ is a finite subfamily for $f^{-1}\beta_n$ and $\bigcup_{n\in N} f^{-1}\delta_n$ is a cover of the spaces (X,U). Note that for any $n \in N$ $f^{-1}\delta_n \wedge \gamma_n = \mu_n$ is a finite subfamily of α_n and $\bigcup_{n\in N}\mu_n$ is a cover of (X,U). Consequently, (X,U) is a uniformly Menger space.

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MSC 34E05

DYNAMICAL SYSTEMS WITH MATRICES OF COEFFICIENTS WITH DOMINANCE

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Supra, the author proved existence of “special“ solutions of initial value problems for dynamical systems: scalar differential equations with small delay of argument and corresponding difference equations. In the paper existence of “special“ solutions is proven for 2-vector differential equations with small delay of argument and corresponding systems of difference equations is proven.
Keywords: finite-difference equation; differential equation with retarded argument; system of equations; special solution; initial problem; asymptotic. dynamical system, solution, differential equation with delay, difference equation, initial value problem.

Мурда автор төмөнкүдөй динамикалык системалар үчүн баштапкы маселелердин «атайын» чыгарылыштарынын жашоосун далилдеди: скалярдык, аргументинин кечигүүсү кичине болгон дифференциалдык теңдемелер менен дал келген айырма теңдемелери. Бул макалада «атайын» чыгарылыштарынын жашоосу 2-вектордук аргументинин кечигүүсү кичине болгон дифференциалдык теңдемелер менен дал келген айырма теңдемелери системалары үчүн далилденди.
Урунттуу сөздөр: динамикалык система, чыгарылыш, кечигүүчү аргументтүү дифференциалдык теңдеме, айырма теңдемеси, баштапкы маселе.

Ранее автор доказала существование «специальных» решений начальных задач для динамических систем: скалярных дифференциальных уравнений с малым запаздыванием аргумента и соответствующих разностных уравнений. В статье доказано существование «специальных» решений для 2-векторных дифференциальных уравнений с малым запаздыванием аргумента и соответствующих систем разностных уравнений.

Ключевые слова: динамическая система, решение, дифференциальное уравнение с запаздывающим аргументом, разностное уравнение, начальная задача.

1. Introduction

Formerly, in order to conduct the in-depth study of differential equations with delay, the author proposed the method of splitting the solution space reducing such equations to the systems of operator-difference equations. Using this method, the author assumed new conditions, i.e. the absolute domains for coefficients sufficient for the existence of special (slowly changing) solutions, and proved the presence of approximating and asymptotically approximating properties in them, as well as the asymptotic one-dimensional space of solutions of the initial problems for linear scalar differential equations with small delay of argument and the corresponding difference equation systems (special solutions correspond, to the solutions with a slowly changing first component and a relatively small second component). For the purposes of the single-point representation of the obtained results and other data related to the theory of dynamic systems (the distance between the solution values tends to zero alongside the unlimited increase in argument), throughout this research paper the author uses the concept of the asymptotic equivalence of solutions for dynamic systems, as it was introduced by the author in their previous papers.

This paper proves existence of “special“ solutions is proven for 2-vector differential equations with small delay of argument and corresponding systems of difference equations.

In order to provide solution of the initial problem for a linear differential equation with a limited retarded argument, some sources, including the book [1] and research papers [2] and [3], discuss the conditions when there exists such one-dimensional sub-space of solutions (referred to as special), so that any solution tends to increase its argument towards one of the special solutions. The review of such academic sources is given in the research paper [4].

The book [5], it is illustrated that using the differential equations with small argument retardation and the technique of splitting the solution space, it is possible to enable their transformation into the systems of operator-difference equations with preservation of their specifics. Thus, the new conditions ensuring the existence of special (slowly changing) solutions are obtained, their possession of approximating and asymptotically approximating properties is defined, as well as the asymptotic one-dimensional solution space of initial problems for linear scalar differential equations with small argument retardation and corresponding difference system of equations (special solutions correspond to the solutions with a slowly changing first component, and a relatively small second component).

Section 2 introduces the required definitions and denotations.

Section 3 presents the theorem on existence of special solutions of systems of difference equations.

Section 4 presents the newly found conditions for 2-vector differential equations with retarded argument.

2. Denotatons and definitions

Let us denote as follows: $\boldsymbol{N}_0 = \{0,1,2,3,\ldots\}$, $\boldsymbol{N} = \{1,2,3,\ldots\}$, $\boldsymbol{R} = (-\infty,\infty)$, $\boldsymbol{R}_+=[0,\infty)$, $\boldsymbol{R}_{++}=(0,\infty)$, I_n- $n\times n$ as identity matrix, $n\in\boldsymbol{N}$; $C^{m(k)}D$ is the space of functions $u: D\to\boldsymbol{R}^m$ continuous with the derivatives till the order k; D is the domain in $\boldsymbol{R}$, $0\in D$, $m\in\boldsymbol{N}$, $k\in\boldsymbol{N}_0$; $C^{*m(k)}D$ is the sub-space of functions meeting the condition $u(0)=0\in\boldsymbol{R}^m$. $m=1$ and $k=0$ shall be neglected.

For the purposes of the uniform representation of problems with continuous and discrete time, it shall be assumed that the argument of the desired function t belongs to quite an ordered set Λ, possessing its smallest element (herein referred to as 0), but still not possessing any largest element. Normally, $\Lambda=\boldsymbol{R}_+$ or $\Lambda=\boldsymbol{N}_0$ are employed.

This paper discusses the initial problems only. Assuming that the initial problem always has some solution, then the solution is the only and global one, and, thus, it covers the entire set Λ; so, then the space of solutions of any particular dynamic system with the entry condition φ may be regarded as the operator $W(t,\varphi):\Lambda\times\Phi\to Z$, with Φ being the topological space of the initial conditions, and Z being the topological space of the solution values. In the case of $\Lambda=\boldsymbol{R}_{+}$, let us assume that $W(t,\varphi)$ is continuous with respect to t.

The following types of space shall be discussed herein, i. e. Φ and Z: linear one-dimensional ($\boldsymbol{R}$); linear multidimensional ($\boldsymbol{R}^d$); Banach.

Definition 2. The following relation of equivalence to space Φ is said to be asymptotic equivalence: If Z is a Banach space, then

$$(\varphi_1 \sim \varphi_2) \Leftrightarrow (\lim\{ ||W(t,\varphi_1)-W(t,\varphi_2)||_Z: t\to\infty\}= 0).$$

Exponential functions are commonly used as comparison functions. Consequently, we introduced

Definition 3. The following relation of equivalence to space Φ is said to be λ-exponential asymptotic equivalence $(\lambda\in\boldsymbol{R}_{++})$: If Z is a Banach space, then

$$(\varphi_1 \sim \varphi_2) \Leftrightarrow (\sup\{ ||W(t,\varphi_1)-W(t,\varphi_2)||_Z \exp(\lambda t): t\in\Lambda\}<\infty).$$

Definition 4. Solutions to $W(t,\varphi)$, $\varphi\in\Phi_0$, are referred to as *special* solutions, if

1) $(\forall t_1\in\Lambda)\ (\forall z_1\in Z)\ (\exists\varphi_1\in\Phi_0)(W(t_1,\varphi_1)=z_1);$

2) $(\exists\lambda > 0)(\forall\varphi_1,\varphi_2\in\Phi_0)(\forall t_1,t_2\in\Lambda)$

$$(||W(t_1,\varphi_1) - W(t_1,\varphi_2)|| \geq ||W(t_2,\varphi_1) - W(t_2,\varphi_2)||\exp(-\lambda|t_1 - t_2|)).$$

Consequently, space Φ_0 for special functions possesses dimensions of space Z (as opposed to the entire space Φ).

3. System of difference equations

Consider the system of difference equations

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 1+a_{11,n} & a_{12,n} & a_{13,n} \\ a_{21,n} & 1+a_{11,n} & a_{23,n} \\ a_{31,n} & a_{32,n} & a_{33,n} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}, \quad n = 0,1,2,... \qquad (1)$$

with the initial condition $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x_{0,0} \\ y_{0,0} \\ z_{0,0} \end{pmatrix}$ where x_n and y_n are scalars and z_n are elements of a Banach space B, $a_{ij,n}$ are corresponding linear operators. Denote $\mathrm{d} := \sup\{||a_{ij,n}||: i, j, n\}$.

Define the norm *max* in the space $R^2 \times B$.

Theorem 1. If $d \leq 3 - \sqrt{8}$ then there exist such (special) solutions $\{X_n, Y_n, Z_n\}$ to the system in (1), so that

$$(\forall n \in N)(\ ||\{X_n, Y_n\}|| \geq \eta_-^{\,n};\ ||Z_n|| \leq w||\{X_n, Y_n\}||). \tag{2}$$

Proof. Let $||\{X_{0,0}, Y_{0,0}\}||=1,\ Z_{0,0}=0.$

For brevity of writing, we will consider pass from $n=0$ to $n=1$.

Without loss of generality, $X_0=1,\ |Y_0| \leq 1$.

$$X_1 \geq 1 \cdot X_0 - |a_{11}|\, X_0 - |a_{12}| \cdot |Y_0| - ||a_{13}|| \cdot ||Z_0|| \geq 1 \cdot 1 - d \cdot 1 - d \cdot 1 - d \cdot w = 1-(2+w)d;$$

$$||Z_1|| \leq |a_{31}|\, X_0 + |a_{32}| \cdot |Y_0| + ||a_{33}|| \cdot ||Z_0|| \leq d \cdot 1 + d \cdot 1 + d \cdot w = (2+w)d.$$

It must be $1-(2+w)d \geq \eta_-$; $(2+w)d \leq w\eta_-$. Hence, $(1-(2+w)d)w \geq (2+w)d.$

$$dw^2+(3d-1)w+2d \leq 0.\ (3d-1)^2-8d^2=0.\ d^2-6d+1=0.$$

$$d_0=3-\sqrt{(9-1)}=3-2.824\ldots=0.175\ldots$$

Theorem is proven.

Also calculate the constants w_0: $w^2+(3-(3+\sqrt{8}))w+2=0.$

$$w^2-\sqrt{8}\,w+2=0.\ w_0=\sqrt{2}.$$

$$\eta_{-0}=1-(2+w_0)d_0=1-(2+\sqrt{2})(3-2\sqrt{2})=1-6-3\sqrt{2}+4\sqrt{2}+4=\sqrt{2}-1.$$

4. 2-vector differential equation with retarded argument

Let us consider an equation:

$$w'(t)=P(t)w(t-h)\ (h \in \mathbf{R}_{++}),\ t \in \mathbf{R}_+, \tag{3}$$

where $w(t)$ is the unknown 2-vector-function, and $P(t)$ is the given continuous 2×2-matrix-function respectively, with the initial condition:

$$w(t)=\varphi(t),\ t \in [-h,0], \tag{4}$$

where $\varphi(t)$ is the given continuous 2-vector-function.

It is well-known that the initial problem (3)-(4) is equivalent to the following system of equations for functions $U_n(t)\in C^{2(1)}[-h,0]$, $n\in \mathbf{N}_0$:

$$U_0(t) = S_0(\varphi(\cdot))(t) := \varphi(0) + \int_{-h}^{t} P(s+h)\varphi(s)ds;$$

$$U_{n+1}(t) = S_n(U_n(\cdot))(t) := U_n(0) + \int_{-h}^{t} P(s+nh+h)U_n(s)ds,\, n\in \mathbf{N}_0. \quad (5)$$

In [7], the author proposed the following method: let us consider the space $C^{2(1)}[-h,0]=\mathbf{R}^2\times C^{2(1)}[-h,0]$ as Cartesian product of space of functions-constants equal to $\mathbf{R}^2$, and the space $\Omega=C^{*2(1)}[-h,0]$, namely $U_n(t)=x_n+y_n(t)$.

Let us denote that

$$P_n(\tau) = P(\tau + nh + h).$$

Consequently, the shift operators equal to the value of h in (5) shall be given in the following way:

$$u_{n+1}+z_{n+1}(t) = S_n(u_n + z_n(\cdot))(t) = u_n + z_n(0) + \int_{-h}^{t} P_n(s)(u_n + z_n(s))ds =$$

$$= u_n + \int_{-h}^{0} P_n(s)(u_n + z_n(s))ds + \int_{0}^{t} P_n(s)(u_n + z_n(s))ds =$$

$$= \left(\int_{-h}^{0} P_n(s)ds + I_2\right)u_n + \int_{-h}^{0} P_n(s)z_n(s)ds + \int_{0}^{t} P_n(s)ds\, u_n + \int_{0}^{t} P_n(s)z_n(s)ds. \quad (6)$$

Let us denote (a 2×2-matrix) $a_n \coloneqq \int_{-h}^{0} P_n(s)ds + I_2$ (vector functional) $b_n z(\cdot) \coloneqq \int_{-h}^{0} P_n(s)z(s)ds$, (2×2-matrix-function) $c_n(t) \coloneqq \int_{0}^{t} P_n(s)ds$,

(operator of the function) $d_n z(\cdot)(t) \coloneqq \int_{0}^{t} P_n(s)z(s)ds$. (7)

Then, the system in (6) is modified as follows:

$$u_{n+1} = a_n u_n + b_n z_n(\cdot),$$

$$z_{n+1}(t) = c_n(t)u_n + d_n z_n(\cdot)(t), n \in N_0. \quad (8)$$

Presenting $P_n(t) = \begin{pmatrix} p_{11,n}(t) & p_{12,n}(t) \\ p_{21,n}(t) & p_{22,n}(t) \end{pmatrix}$ and $u_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ we obtain a system of type (1).

Theorem 2. If $\Delta := \max\{\sup\{|p_{ij,n}(t)|: t \in R_+\}: i,j = 1,2\}h$ is sufficiently small then the equation (3) has special solutions:

$$W(0) \neq 0, \ ||W(nh)|| \leq \eta_-^n ||W(0)||, n \in N.$$

Proof (briefly). Norms of all operators in (7) are estimated by Δ. Applying of Theorem 1 completes the proof.

Conclusion

A new class of matrix-functions "with dominance" is distinguished. Some elements of the main diagonal are close to one, others are small in norm. Such matrix-functions provide category of dynamical systems with special solutions.

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MSC 34A26, 35A16

OPERATIONS OVER FUNCTIONAL RELATIONS

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Solutions of some ordinary and partial differential equations have values in some points connected by functional relations: even, odd and periodical solutions (2 points), Asgeirsson's identity for partial differential equations of hyperbolic type (4 points), Vallée-Poussin's assertion, Lagrange interpolation polynomial (many points). In the paper operations over functional relations are proposed and investigated.

Keywords: functional relation, ordinary differential equation, partial differential equation, solution, operation, algorithm.

Ар кандай типтеги кадимки жана жеке туундулуу дифференциалдык теңдемелердин чыгырылыштарында кээ бир чекиттеринде функционалдык өз ара байланыштар маанилер бар. Мисалы, жуп, так жана мезгилдүү чыгарылыштар (2 чекит), Асгейрссон бирдейлиги (4 чекит), Валле-Пуссен катышы, кадимки дифференциалдык теңдемелер үчүн Лагранждын интерполяциялык полиному (көп чекит). Макалада функционалдык өз ара байланыштардын үстүндө амалдар сунушталат жана изилделет.

Урунттуу сөздөр: функционалдык өз ара байланыш, кадимки дифференциалдык теңдеме, жеке туундулуу дифференциалдык теңдеме, чыгарылыш, амал, алгоритм.

Решения некоторых обыкновенных дифференциальных уравнений и уравнений в частных производных имеют значения в некоторых точках, связанные функциональными соотношениями: четные, нечетные и периодические решения (2 точки), тождество Асгейрссона (4 точки), соотношение Валле-Пуссена, интерполяционный многочлен Лагранжа (много точек). В статье рассматриваются и исследуются операции над функциональными соотношениями.

Ключевые слова: функциональное соотношение, обыкновенное дифференциальное уравнение, дифференциальное уравнение в частных производных, решение, операция, алгоритм.

1. Introduction

We consider the following fact: solutions of some types of differential equations have functional relations connecting their values in different points. By given values of solutions in several points one can find their values in other points.

As we know, the first example of sufficient functional relation was [1].

In [2-6] we considered some aspects of functional relations.

Remark. Survey of papers [7-19] and other papers demonstrate that there was not any general classification of differential equations. Functional relations yield such possibility.

In the paper we propose and investigate operations over functional relations.

The second section contains definitions and lists various operations over functional relations.

The third section contains examples of application of operations.

In this paper we will use functional denotations of type $x[n]$ instead of x_n.

2. Functional relations and operations with them

A functional relation in the most general case can be written as follows.

Let Q be a set of some subsets of the set X.

Definition 1. A function $g: W\rightarrow Z$ where $W\in Q$ is said to be a functional element for the sets X and Z and family of sets Q.

Definition 2. Let P be a predicate defined on functional elements of the triple (X, Z, Q). If all restrictions of a function $f:X\rightarrow Z$ on $W\in Q$ fulfill P then it is said that the function f fulfill the functional relation P.

Besides well-known logical operations extended to predicates P, there can be defined specific operations over functional relations.

Operations of the first kind.

A binary operation on a functional relation P can be written in general form as a partially defined operator $\Lambda:Q\times Q\rightarrow Q$ which constructs the set $\Lambda(W_1,W_2)$ as a subset of $W_1\cup W_2$.

Such operations transform the set Q into itself.

Operations of the second kind.

A binary operation on a functional relation P can be written in general form as a partially defined operator $\Lambda:Q\times Q\rightarrow Q_2$ which generates a new functional relation of the set Q_2 and corresponding predicate P_2 which are logical consequences a functional relation P. The set $\Lambda(W_1,W_2)$ is also a subset of $W_1\cup W_2$ but does not belongs to Q.

Such operations generate new functional relations.

2.1. Adding of formulas

2.2. Multiplying of formulas

2.3. Conjunction of predicates

2.4. Continuation for finite sets W. Some predicates meet the condition:

If $(x[1], x[2], \dots, x[n-1], x[n]) \in Q$ and $(x[2], x[3], \dots, x[n-1], x[n], x[n+1]) \in Q$ then $(x[1], x[3], \dots, x[n-1], x[n], x[n+1]) \in Q$.

Thus, we obtain the operation of the first kind:

$$\Lambda\,((x[1], x[2], \dots, x[n-1], x[n]),(x[2], x[3], \dots, x[n-1], x[n], x[n+1])) =$$
$$= (x[1], x[3], \dots, x[n-1], x[n], x[n+1]).$$

2. 5. Adhesion for sets W of even number of members.

If $(x[1], x[2], \dots, x[n], x[n+1], \dots, x[2n-1], x[2n]) \in Q$ and

$(x[n+1], \dots, x[2n, x[2n+1], \dots, x[3n]) \in Q$

then consider the set $(x[1], x[2], \dots, x[n-1], x[n], x[2n+1], \dots, x[3n])$.

Thus, we obtain the operation which can be both of the first kind and of the second kind:

$$\Lambda\,((x[1], x[2], \dots, x[n], x[n+1], \dots, x[2n-1], x[2n]),$$
$$(x[n+1], \dots, x[2n], x[2n+1], \dots, x[3n])) =$$
$$= (x[1], x[2], \dots, x[n-1], x[n], x[2n+1], \dots, x[3n]).$$

3. Examples of functional relations with operations

Denote the functional relation number F for every equation as the minimal number of connected points (if it exists). We will give either mention of k-point value problem or a formula (*).

3.1. Odd and even functions on R. $F=2$, $X_1 =\{x \in R | (\exists x_0 \in R)(|x|=|x_0|)$.

Odd functions: $f(-x)= -f(x)$.

Even functions: $f(-x)= f(x)$.

(See 2.2). Squaring we obtain: $f^2(-x)= f^2(x)$ both for odd and even functions.

3.2. The set of IVP $y'(x)=a$, $y(0)=0$, arbitrary $a\neq 0$: $F=2$:

$$(*)\ y(x[1])x[2] - y(x[2])x[1]=0.$$

Adding the assertion $y(x[1])x[3] - y(x[3])x[1]=0$ and subtracting we obtain

$$y(x[1])(x[2]-x[3])-(y(x[2])-y(x[3]))x[1]=0,$$

$$(y(x[2])-y(x[3]))/(x[2]-x[3])=const.$$

It is another definition of the set of solutions of the IVP.

3.3. The set of IVP: $y'(x)=a,\ y(0)=y_0$, arbitrary $a\neq 0,\ y_0;\ F=3:$

$$(*)\ (y(x[1])-y(x[3]))(x[1]-x[2])-(y(x[1])-y(x[2]))(x[1]-x[3])=0.$$

There is an analogous result.

2.4. The linear differential equation of the k-th order $y^{(k)}(x)=0$, or a polynomial of $(k-1)$-th order: $F=k+1$. Let numbers $x[1]< x[2]<\ldots <x[k+1],\ y[1],\ y[2],\ldots,y[k+1]$ be given. Construct the Lagrange interpolation polynomial of the $(k-1)$-th order by the values $x[1],\ x[2],\ldots,x[k]$ и $y[1],\ y[2],\ldots,y[k]$ then (*) $L(x[k+1])-y[k+1]=0.$

(See 2.4). If only $x[k+1]-x[1]< d$ (a small number) is permitted, then step-by-step the polynomial can be constructed for any set of $\{x(j)\}$.

2.5. A solution of the hyperbolic equation $\frac{\partial^2}{\partial x_1\,\partial x_2}u(x_1,x_2)=0$ meets the Asgeirsson's identity $(F=4)$:

$$(*)\ u(w[1],v[1])+u(w[2],v[2])-u(w[1],v[2])-u(w[2],v[1])\equiv 0.$$

(See 2.1) Consider the following identity:

$$(**)\ u(w[2],v[1])+u(w[3],v[2])-u(w[2],v[2])-u(w[3],v[1])\equiv 0.$$

Adding (**) to (*) we obtain

$$u(w[1],v[1])+u(w[3],v[2])-u(w[1],v[2])-u(w[3],v[1])\equiv 0.$$

If in (*) only $|w[2]-w[1]|+|v[2]-v[1]|<d$ (a small number) is permitted, then step-by-step any point can be reached.

2.6. A solution of the wave equation $\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)=\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)$ meets the similar Asgeirsson's identity $(F=4)$: for four vertices of a rectangle obtained by means of rotation of the rectangle (*) on 45^0.

4. Conclusion

We hope that development of operations on functional relations would yield new methods to solve various differential equations approximately and obtain new information on differential equations as whole. Some techniques mentioned can be implemented in any algorithmic language.

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MSC 45A05, 45G10, 47H30

CATEGORY OF CORRECT EQUATIONS CONTAINS MULTI-DIMENSIONAL INTEGRAL EQUATIONS OF THE FIRST KIND

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Supra, the author participated in introduction of elements of category of equations on the base of the principle of preservation of solution while transformations and of its subcategories including the category of correct equations, and proved correctness of some integral equations of the first kind with one and two arguments. In this paper correctness of some multi-dimensional integral equations of the first kind is proven.

Keywords: category, solution, integral equation of the first kind, correctness, analytical function.

Мурда автор өзгөртүүлөрдү киргизүүдө чыгарылышты сактоо принцибинин негизинде теңдеме категориясынын элементтерин жана анын жарым-категорияларын, ошондой эле корректтүү теңдемелердин категориясын дагы киргизүүгө катышкан, жана бир жана эки аргументтүү биринчи түрдөгү кээ бир интегралдык теңдемелердин корректтүүлүгүн далилдеген. Бул макалада кээ бир аналитикалык функциялуу биринчи түрдөгү көп өлчөмдүү интегралдык теңдемелердин корректүүлүгү далилденген.
Урунттуу сөздөр: категория, чыгарылыш, биринчи түрүндөгү интегралдык теңдеме, корректтүүлүк, аналитикалык функция.

Ранее автор участвовала во введении элементов категории уравнений на основе принципа сохранения решения при преобразованиях и ее подкатегорий, в том числе категории корректных уравнений, и доказала корректность некоторых интегральных уравнений первого рода с одним и двумя аргументами. В данной статье доказана корректность некоторых многомерных интегральных уравнений первого рода с аналитическими функциями.
Ключевые слова: категория, решение, интегральное уравнение первого рода, корректность, аналитическая функция.

1. Introduction

The approach of categories as notions being more general than sets and families of sets was introduced in [1] and was used firstly in topology and general algebra. In Kyrgyzstan the first works on the category theory were [2] and [3].

The author participated in introduction of elements of category of equations [4], [5]. A new general notion of equation was introduced by us with assistance of the notion “predicate” on the base of the principle of preservation of solution while transformations (supra it had been meant implicitly) and elements of the category of equations were constructed on the base of well-known categories. Further, the author participated in construction of its subcategories of the category of equations including the category of correct equations [6], [7]. This notion included the known “correctness by Hadamard”.

The author proved correctness of some integral equations of the first kind [8], [9]: Solutions of a two-dimensional integral equation of the first kind with a kernel being an exponentially-quadratic-decreasing, depending on difference of arguments function exists and depends on right hand part continuously in the space of analytical functions of exponential type.

In this paper correctness of some multi-dimensional integral equations of the first kind is proven.

2. Survey of known results

The main categories are the following:

The category of sets *Set*. Objects *Ob(Set)* are non-empty sets, morphisms *Mor(Set)* are functions.

The category of functions *Func*. *Ob(Func) = Mor(Set), Mor(Func)* are transformations of functions.

These categories are used in developing of the category of equations *Equa*.

The category of topological spaces *Top*. *Ob(Top)*are topological spaces, *Mor(Top)* are continuous functions.

This category is used in developing of the subcategory *Equa-Par-Top*.

Supra equations were subdivided informally. We used the fact that equations and systems of equations of various types are equivalent. Moreover, the well known technique of reducing order of differential equations, various techniques of substitution and transforming of argument, the method of transforming of solutions developed in Kyrgyzstan, the method of additional argument and the method of reducing differential equations to systems of operator-differential ones developed in Kyrgyzstan demonstrated that equations with various solutions and even in various spaces can be equivalent.

Hence, we enlarged the notion of equation including «systems of equations», «with initial or boundary conditions» to formulate main notions, objects and morphisms of the category *Equa* of equations and its subcategories.

Definition 1. *Ob(Equa)* contains the following tuples

{Non-empty sets $X, Y;$ predicate $P(x)$ on $X;$ transformation $B : X \to Y\}.$

If $(\exists x \in X)(P(x) \wedge (y=B(x)))$ then $y \in Y$ is said to be a solution of the equation $\{X, Y, P, B\}$. Particularly, if B is the identity operator I, then we obtain the equation *“P(x)”*only.

Mor(Equa) are such transformations of tuples *{X, Y, P, B}* that solutions (or their absence) preserve.

Some subcategories for the category *Equa*.

The category of equations for functions *Equa-Func*.

Definition 2. *Ob(Equa-Func)* contains the following tuples

$\{X \in Ob(Func), Y \in Ob(Func),$ predicate $P(x)$ on X, transformation $B{:}X \to Y\}$.

Mor(Equa-Func) contains invertible transformations of functions inherited from *Mor(Equa)* and specific transformations.

For the category of equations with parameters we proposed

Definition 3. *Ob(Equa-Par)* contains the following tuples

$\{$non-empty sets $X, F, Y,$ predicate $P(x,f)$ on $X \times F$, transformation $B{:}X \to Y\}$.

If $(\exists x \in X)(P(x,f) \wedge (y=B(x))$ then then $y \in Y$ is said to be a solution of the equation $\{X, F, Y, P, B\}$.

Mor(Equa-Par) are such transformations of tuples *{X, Y, P, B}* (except *F*) that solutions (or their absence) preserve.

By our approach «correctness» can by a parameter only, hence the category of correct equations *Equa-Par-Top* is a subcategory of the category *Equa-Par*.

Definition 4. *Ob(Equa-Par-Top)* are tuples

$\{$ topological spaces $X, F, Y;$

predicate $P(x,f)$ on $X \times F;$

continuous transformation $B{:}X \to Y \}$

such that 1) $(\forall f \in F)(\exists! y \in Y)(\exists x \in X)\ (P(x,f) \wedge (y=B(x));$

2) the element y depends on the element f continuously.

Mor(Equa-Par-Top) are transformations preserving properties 1) and 2).

The category *Equa-Func-Par-Top* of correct equations for functions also can be defined.

We considered the integral operator

$$K(x; w(s){:}\, s) := \int_{-\infty}^{\infty} \exp(-(x-s))w(s)ds \qquad (1)$$

in the space A_{+v} of entire analytical functions with the norm

$$|| f ||_{v} := sup\{| f^{(n)}(0)|v^{-n} : n = 0,1,2,...\}, v > 0.$$

We proved that the solution of the equation $K(x; w(s): s) = f(x)$ exists and can be presented as

$$w(x) = K^{-1}(x; f(s): s) = \sqrt{\pi^{-1}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{1}{4} \right)^{k} f^{(2k)}(x) =$$

$$= \sqrt{\pi^{-1}} \exp\left(-\frac{1}{4} \frac{d^2}{dx^2} \right) f(x). \qquad (2)$$

We extended such phenomenon for certain linear and non-linear integral equations.

We considered the integral operator

$$K_2(x, y; w(s,u): s, u) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-(x-s)-(y-u)) w(s,u) ds du \qquad (3)$$

in the space $A_{2.+v}$ of entire analytical functions in both variables with the norm

$$||f||_{v} := sup\{| \frac{\partial^{p+q}}{\partial^{p} x \partial^{q} y} f(0,0)| v^{-p-q} : p, q = 0,1,2, \ldots\}, v > 0.$$

We proved that the solution of the equation $K_2(x, y; w(s,u): s, u) = f(x, y)$ exists and can be presented as

$$w(x,y) = K_2{}^{-1}(x, y; f(s,u): s, u) = \pi^{-1} exp\left(-\frac{1}{4} \frac{\partial^2}{\partial^2 x} - \frac{1}{4} \frac{\partial^2}{\partial^2 y} \right) f(x,y). \quad (4)$$

3. Multi-dimensional integral equation of the first kind

Consider the equation

$$J_n(x_1, x_2, \ldots, x_n; w(s_1, s_2, \ldots, s_n): s_1, s_2, \ldots, s_n) :=$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\left(-\sum_{k=1}^{n} (x_k - \xi_k)^2 \right) w(\xi_1, \xi_2, \ldots, \xi_n) d\xi_1 d\xi_2 \ldots d\xi_n =$$

$$= f(x_1, x_2, \ldots, x_n). \qquad (5)$$

To investigate this equation we also consider the multi-dimensional partial differential heat equation with reverse time in $\boldsymbol{R}^n$:

$$\frac{\partial u(t,x_1,x_2,\ldots,x_n)}{\partial t} = -a\Delta u(t,x_1,x_2,\ldots,x_n), (t,x_1,x_2,\ldots,x_n) \in \boldsymbol{R}_{++} \times \boldsymbol{R^n},\ a > 0 \quad (6)$$

with the initial condition

$$u(0,x_1,x_2,\ldots,x_n) = \varphi(x_1,x_2,\ldots,x_n), (x_1,x_2,\ldots,x_n) \in \boldsymbol{R^n}. \quad (7)$$

Let

$$\varphi \in A_{n,+v} \quad (8)$$

where $A_{n,+v}$ is the space of entire analytical functions with real coefficients with the condition for the multi-index $(\forall \boldsymbol{I} \geq \boldsymbol{0})\left(\left|\frac{\partial^{|I|} f(0)}{\partial x^I}\right| < const \cdot v^{|I|}\right)$ and the norm

$$||\boldsymbol{f}(\boldsymbol{x})||_{\boldsymbol{n+v}} \coloneqq \boldsymbol{sup}\left\{\left|\frac{\partial^{|I|} f(0)}{\partial x^I}\right| v^{-|I|} : I \in N_0^n\right\}.$$

Theorem 1. If (8) fulfils then there exist an entire analytical solution of the problem (6)-(7) by the formula:

$$u(t,x_1,x_2,\ldots,x_n) = \exp(-at\Delta)\varphi(x_1,x_2,\ldots,x_n). \quad (9)$$

Proof. The series (9) fulfills (6)-(7) formally. It is majored by the converging series

$$\sum_{k=0}^{\infty} \frac{1}{k!} a^k \left\| \Delta^k \varphi(x_1,x_2,\ldots,x_n) \right\|_{n,+v} t^k \leq$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{k!} a^k (nv^2)^k \left\| \varphi(x_1,x_2,\ldots,x_n) \right\|_{n,+v} t^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (atnv^2)^k \left\| \varphi(x_1,x_2,\ldots,x_n) \right\|_{n,+v} t^k = \left\| \varphi(x) \right\|_{n,+v} \exp(anv^2 t).$$

Corollary. If (8) fulfils then the integral equation (5) has a stable solution.

Proof. The solution is given by the formula

$$J_n{}^{-1}(x_1,x_2,\ldots,x_n; f(s_1,s_2,\ldots,s_n): s_1,s_2,\ldots,s_n) =$$

$$= \pi^{-\frac{n}{2}} \sum_{k=0}^{\infty} \frac{1}{k!4^k} (-1)^k \Delta^k f(x_1,x_2,\ldots,x_n).$$

By means of various transformations of (5) other multi-dimensional integral equations of the first kind belonging to the category of correct equations can be obtained.

Conclusion

We have proven correctness of the integral equation of the first kind (5). We hope that the category of correct equations can be further enlarged by the proposed method.

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MSC 34K20, 45J05

ON A METHOD FOR INVESTIGATING THE STABILITY OF SOLUTIONS OF A SYSTEM OF THIRD-ORDER LINEAR INTEGRO-DIFFERENTIAL EQUATIONS OF THE VOLTERRA TYPE

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Sufficient boundedness conditions are established on the half-axis of all solutions and their first and second derivatives, i.e. stability conditions of solution of the system of third order Volterra integro-differential equations. A method of splitting into systems of equations of the first and second orders has developed.

Keywords: integro-differential equations of third order, system, boundedness of solutions and their derivatives, stability, splitting method.

Вольтерра тибиндеги сызыктуу үчүнчү тартиптеги интегро-дифференциалдык теңдемелер системасынын бардык чыгарылыштарынын жана алардын биринчи жана экинчи туундуларынын жарым окто чектелгендигинин, б.а. чыгарылыштарынын турумдуулугунун жетиштүү шарттары табылат. Үчүнчү тартиптеги системаны биринчи жана экинчи тартиптеги системаларга ажыратуу методу иштелип чыгат.

Урунттуу сөздөр: үчүнчү тартиптеги интегро дифференциалдык теңдемелер, система, чыгарылыштардын жана алардын туундуларынын чектелгендиги, турумдуулук, ажыратуу методу.

Устанавливаются достаточные условия ограниченности на полуоси всех решений и их первых, и вторых производных, т.е. устойчивости решений линейной системы интегро-дифференциальных уравнений третьего порядка типа Вольтерра. Разработан метод расщепления на системы уравнений первого и второго порядков.

Ключевые слова: интегро-дифференциальные уравнения третьего порядка, система, ограниченность решений и их производных, устойчивость, метод расщепления.

All the functions involved and their derivatives, the relations are valid for $t \geq t_0, t \geq \tau \geq t_0, J = [t_0, \infty)$; SIDE is a system of integro-differential equations; $\langle a, b \rangle$ scalar product of $n \times 1$ vectors a, b; relations for matrices are understood as relations for their quadratic forms with any non-zero vector; A^T is the transposed matrix A; $\mathrm{E} - n \times n$ the unit matrix; under $\|x\|$ for $n\times 1$ vector and $\|A\|$ for $n\times n$ matrices $A=(a_{ij})$ understood [1, c.35], accordingly,

$$\|x\|=\left(\sum_{i=1}^{n} x_i^{\,2}\right)^{\frac{1}{2}} \text{ и } \|A\|=\sqrt{\gamma},$$

where $\gamma=\max\lambda_i\,(1\le i\le n);\lambda_i$ - eigenvalues of the symmetric matrix $A^T A$. Note that the relation holds [2, c.10]:

$$\|A\|\le\left(\sum_{i,j=1}^{n} a_{ij}^{\,2}\right)^{\frac{1}{2}}.$$

The stability of solutions of a third-order linear SIDE is understood as the boundedness on the half-interval J of all its solutions and their first and second derivatives.

PROBLEM. To establish sufficient conditions for the stability of solutions of an n - dimensional third-order SIDE of the Volterra type of the form

$$x'''(t)+A_2(t)x''(t)+A_1(t)x'(t)+A_0(t)x(t)+$$
$$+\int_{t_0}^{t}\left[Q_0(t,\tau)x(\tau)+Q_1(t,\tau)x'(\tau)+Q_2(t,\tau)x''(\tau)\right]d\tau=f(t),t\ge t_0, \quad (1)$$

where $x(t)=(x_i(t))$ is unknown $n\times 1$ vector function; $A_k(t),Q_k(t,\tau)(k=0,1,2)$ is known $n\times n$ matrix functions; $f(t)$ is known $n\times 1$ vector function; at the same time , as you know, $A_k(t)(k=0,1,2)$ are called coefficients, $Q_k(t,\tau)(k=0,1,2)$- kernels, $f(t)$ is a free member.

Note that earlier such a problem was solved in the article [3] by a modified method of transformation of equations [4, pp.28-29]. In this article, the problem is solved by the development of a non-standard method of reduction to a system of [5], the matrix method of cutting functions [6], the method of transformation of V.Volterra equations [7, pp.194-217] and the method of integral inequalities [8].

Let's start getting the main result.

In SIDE (1) we will make a non-standard replacement [5]:

$$x'(t)+B(t)x(t)=W(t)y(t), \tag{2}$$

where $B(t),W(t)$-some $n\times n$ weighting matrix functions, with $B(t)\geq 0, W(t)>0; y(t)$ is new unknown $n\times 1$ vector function.

From (2) by differentiation we have

$$\begin{aligned} x''(t) &= -B'(t)x(t)-B(t)x'(t)+W'(t)y(t)+W(t)y'(t)= \\ &= -B'(t)x(t)-B(t)\left[-B(t)x(t)+W(t)y(t)\right]+W'(t)y(t)+ \\ &+W(t)y'(t)=B_*(t)x(t)+W_*(t)y(t)+W(t)y'(t), \end{aligned} \tag{3}$$

where $B_*(t)\equiv B^2(t)-B'(t), W'(t)-B(t)W(t)$.

Further from (3) by differentiating and using substitution (2), we obtain

$$\begin{aligned} x'''(t) &= B_*'(t)x(t)+B_*(t)x'(t)+W_*'(t)y(t)+W_*(t)y'(t)+W'(t)y'(t)+ \\ &+W(t)y''(t)=B_*'(t)x(t)+B_*(t)\left[-B(t)x(t)+W(t)y(t)\right]+W_*'(t)y(t)+ \\ &+\left[W_*(t)+W'(t)\right]y'(t)+W(t)y''(t)=\left[B_*'(t)-B_*(t)B(t)\right]x(t)+ \\ &+\left[W_*'(t)+B_*(t)W(t)\right]y(t)+\left[W_*(t)+W'(t)\right]y'(t)+W(t)y''(t). \end{aligned} \tag{4}$$

Substituting (2)-(4) in SIDE (1), we will have

$$\begin{aligned} &\left[B_*'(t)-B_*(t)B(t)\right]x(t)+\left[W_*'(t)+B_*(t)W(t)\right]y(t)+\left[W_*(t)+W'(t)\right]y'(t)+ \\ &+W(t)y''(t)+A_2(t)\left[B_*(t)x(t)+W_*(t)y(t)+W(t)y'(t)\right]+ \\ &+A_1(t)\left[-B(t)x(t)+W(t)y(t)\right]+A_0(t)x(t)+ \\ &+\int_{t_0}^{t}\Big\{Q_0(t,\tau)x(\tau)+Q_1(t,\tau)\left[-B(\tau)x(\tau)+W(\tau)y(\tau)\right]+ \\ &+Q_2(t,\tau)[B_*(\tau)x(\tau)+W_*(\tau)y(\tau)+W(\tau)y'(\tau)]\Big\}d\tau=f(t), t\geq t_0. \end{aligned} \tag{5}$$

Let 's introduce the notation:

$$\begin{aligned} &B_0(t)\equiv W^{-1}(t)\left[B_*'(t)-B_*(t)B(t)+A_2(t)B_*(t)-A_1(t)B(t)+A_0(t)\right], \\ &B_1(t)\equiv W^{-1}(t)\left[W_*'(t)+B_*(t)W(t)+A_2(t)W_*(t)+A_1(t)W(t)\right], \\ &B_2(t)\equiv W^{-1}(t)\left[W_*(t)+W'(t)+A_2(t)W(t)\right], \\ &P_0(t,\tau)\equiv W^{-1}(t)\left[Q_0(t,\tau)-Q_1(t,\tau)B(\tau)+Q_2(t,\tau)B_*(\tau)\right], \\ &P_1(t,\tau)\equiv W^{-1}(t)\left[Q_1(t,\tau)W(\tau)+Q_2(t,\tau)W_*(\tau)\right], \\ &K(t,\tau)\equiv W^{-1}(t)Q_2(t,\tau)W(\tau), F(t)\equiv W^{-1}(t)f(t). \end{aligned}$$

Taking into account these remarks, multiplying from the left by the matrix $W^{-1}(t)$, from (5) we get a second-order SIDE for $y(t)$, then combining this SIDE with the replacement (2), we come to the next SIDE for $x(t), y(t)$, the equivalent given third-order SIDE (1):

$$\begin{cases} x'(t)+B(t)x(t)=W(t)y(t), \\ y''(t)+B_2(t)y'(t)+B_1(t)y(t)+B_0(t)x(t)+ \\ +\int_{t_0}^{t}\left[P_0(t,\tau)x(\tau)+P_1(t,\tau)y(\tau)+K(t,\tau)y'(\tau)\right]d\tau=F(t), t\ge t_0. \end{cases} \quad (6)$$

Let [6]:

$$K(t,\tau)=\sum_{j=0}^{m}K_j(t,\tau), \quad (K)$$

$$f(t)=\sum_{j=0}^{m}f_j(t), \quad (f)$$

$\Psi_j(t)(j=1..m)$ are some non - singular cutting $n\times n$ matrix functions,

$$\begin{aligned} &R_j(t,\tau)\equiv\left(\Psi_j^{-1}(t)\right)^T K_j(t,\tau)\Psi_j^{-1}(\tau), r_j(t)\equiv\left(\Psi_j^{-1}(t)\right)^T f_j(t), \\ &R_j(t,t_0)=M_j(t)+T_j(t)(j=1,\dots,m), \end{aligned} \quad (R)$$

$c_j(t)(j=1..m)$ are some scalar functions; $R_j(t,\tau)(j=1..m)$ are symmetric $n\times n$ matrices; $B_1(t)$ is symmetric $n\times n$ matrix.

Next, we multiply the first system of (6) scalar by the vector $x(t)$, the second system is on the vector $y'(t)$ and doing the same as in [6], we get the identity:

$$\|x(t)\|^2+2\int_{t_0}^{t}\langle B(s)x(s),x(s)\rangle ds+\|y'(t)\|^2+2\int_{t_0}^{t}\langle B_2(s)y'(s),y'(s)\rangle ds+\langle B_1(t)y(t),y(t)\rangle+$$

$$+\sum_{j=1}^{m}\left\{\begin{array}{l}\langle M_j(t)y_j(t,t_0),y_j(t,t_0)\rangle+\langle T_j(t)y_j(t,t_0),y_j(t,t_0)\rangle-2\langle r_j(t),y_j(t,t_0)\rangle+c_j(t)- \\ -\int_{t_0}^{t}\left[\langle T_j'(s)y_j(t,t_0),y_j(t,t_0)\rangle-2\langle r_j'(s),y_j(s,t_0)\rangle+c_j'(s)\right]ds\end{array}\right\}\equiv$$

$$\equiv c_*+\int_{t_0}^{t}\left[\begin{array}{l}2\langle W(s)y(s),x(s)\rangle+ \\ +\langle B_1'(s)y(s),y(s)\rangle\end{array}\right]ds+$$

$$+\sum_{j=1}^{m}\int_{t_0}^{t}\left[\left\langle M_j{}'(s)y_j(s,t_0),y_j(s,t_0)\right\rangle+\int_{t_0}^{s}\left\langle R_{js\tau}{}''(s,\tau)y_j(s,\tau),y_j(s,\tau)\right\rangle d\tau\right]ds- \qquad (7)$$

where $y_j(t,\tau)\equiv\int_{\tau}^{t}\Psi_j(\eta)x(\eta)d\eta\,(j=1..m),c_*=\|x(t_0)\|+\|y'(t_0)\|+\sum_{j=1}^{m}c_j(t_0).$

Similarly to theorem 1 of [6] is true

Theorem. Let 1) the conditions (*K*), (*f*), (*R*) are true; matrices $B_1(t),R_j(t,\tau)(1..m)$- symmetrical; 2) $B(t)\geq 0;$ 3) $B_2(t)\geq 0;$ 4) $B_1(t)\geq\beta E$, where $0<\beta=const$, there is a scalar function $b_1^*(t)\in L^1(J,R_+)$ such, that $B_1'(t)\leq b_1^*(t)B_1(t);$ 5) $M_j(t)\geq 0,T_j(t)\geq 0,T_j'(t)\leq 0,R_{j\tau}{}'(t,\tau)\geq 0,$ there are scalar functions $\gamma_j(t)\in L^1(J,R_+),c_j(t),\delta_j(t)\in L^1(J,R_+)$ such, that $M_j'(t)\leq\gamma_j(t)M_j(t),$ for any $n\times n$ vectors u_j:

$$(-1)^k\left[\left\langle T_j^{(k)}(t)u_j,u_j\right\rangle-2\left\langle r_j'(t),u_j\right\rangle+c_j^{(k)}(t)\right]\geq 0,R''_{jt\tau}(t,\tau)\leq\delta_j(t)R'_{j\tau}(t,\tau)(k=0,1;\,j=1..m);$$

6) $\|W(t)\|+\|B_0(t)\|+\|f_0(t)\|+\int_{t_0}^{t}\left[\|P_0(t,\tau)\|+\|P_1(t,\tau)\|+\|K_0(t,\tau)\|\right]d\tau\in L^1(J,R_+\setminus\{0\}).$

Then for any solution $(x(t),y(t))$ of SIDE (6) the statements are true:

$$\|x(t)\|=O(1),\left\|y^{(k)}(t)\right\|=O(1)(k=0,1).$$

Let, in addition, 7) $\left\|B^{(k)}(t)\right\|=O(1),\left\|W^{(k)}(t)\right\|=O(1)(k=0,1).$

Then for any solution $x(t)$ of SIDE (1): $\left\|x^{(\nu)}(t)\right\|=O(1)(\nu=0,1,2)$, i.e. any solution of SIDE (1) is stable.

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MSC 34K20, 45J05

ON THE LYAPUNOV FUNCTIONAL FOR THE STABILITY OF SOLUTIONS OF A LINEAR VOLTERRA INTEGRO-DIFFERENTIAL SECOND-ORDER EQUATION ON THE SEMI-AXIS

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Sufficient conditions are established for all solutions and their first derivatives to be bounded on the semiaxis, i.e. stability of solutions of a second-order linear integro-differential equation of Volterra type. A new construction of the Lyapunov method is constructed, i.e. the method of Lyapunov functionals is being developed. An illustrative example is given.

Keywords: linear integro-differential equation of the second order, stability of solutions, sufficient conditions, method of Lyapunov functionals, new construction.

Экинчи тартиптеги сызыктуу Вольтерра тибиндеги интегро-дифференциальдык теңдеменин бардык чыгарылыштарынын жана алардын биринчи туундуларынын жарым окто чектелгендигинин, б.а. чыгарылыштарынын турумдуулугунун жетиштуу шарттары табылат. Ляпуновдун функционалынын жаңы конструкциясы тургузулат, б.а. Ляпуновдун функционалдар методу өнүктүрүлөт. Иллюстративдик мисал келтирилет.

Урунттуу сөздөр: экинчи тартиптеги сызыктуу интегро-дифференциальдык теңдеме, чыгарылыштардын турумдуулугу, жетиштуу шарттар, Ляпуновдун функционалдар методу, жаңы конструкция.

Устанавливаются достаточные условия ограниченности на полуоси всех решений и их первых производных, т.е. устойчивости решений линейного интегро-дифференциального уравнения второго порядка типа Вольтерра. Строится новая конструкция метода Ляпунова, т.е. развивается метод функционалов Ляпунова. Приводится иллюстративный пример.

Ключевые слова: линейные интегро-дифференциальное уравнение второго порядка, устойчивость решений, достаточные условия, метод функционалов Ляпунова, новая конструкция.

All appearing functions and their first derivatives are continuous and the relations take place under the $t \ge t_0, t \ge \tau \ge t_0$; $J = [t_0, \infty)$; IDE – an integro-differential equation.

PROBLEM. Establish sufficient conditions for all solutions and their derivatives to be bounded on the half-interval J, i.e. stability of solutions of a second-order linear integro-differential equation of the Volterra type of the form:

$$x''(t) + a_1(t)x'(t) + a_0(t)x(t) + \int_{t_0}^{t} [K(t,\tau) + C(t,\tau)]x'(\tau)d\tau = f(t), t \ge t_0, \quad (1)$$

where $K(t,\tau)$ is the kernel of the S. Iskandarov – D.N. Shabdanov type [1], $C(t,\tau)$ is the kernel of the T.A.Burton's type [2, p, 216].

As far as we know, no one has studied such a problem before.

In this paper, to solve the problem posed, we propose a construction of the Lyapunov functional using the idea of the partial cut-off method [1] and the idea of constructing the Lyapunov functional from [2, p. 216]. The result is a new construction of the Lyapunov functional.

Let [1, 3]:

$$K(t,\tau) = \sum_{i=1}^{n} K_i(t,\tau), \quad (K)$$

$$f(t)=\sum_{i=1}^{n} f_i(t), \qquad (f)$$

$\psi_i(t)\ (i=1..n)$ are some cutting functions,

$$P_i(t)\equiv K_i(t,t)(\psi_i(t))^{-2},\ \ Q_i(t,\tau)\equiv K_i(t,\tau)(\psi_i(\tau))^{-1},$$

$$E_i(t)\equiv f_i(t)(\psi_i(t))^{-1}\ (i=1..n),\ \ P_i(t)\equiv A_i(t)+B_i(t)\ (i=1..n), \qquad (P)$$

$c_i(t)\ (i=1..n)$ are some functions; similarly to [2, p. 216], the integral $\int_{t_0}^{\infty} C(s,t)ds$ defined for all $t\ge t_0$. This condition for the kernel $C(t,\tau)$ will be called condition (C).

Theorem. Let 1) conditions *(K)*, *(f)*, *(P)*, (C) are satisfied;

2) $\Delta(t)\equiv 2a_1(t)-\int_{t_0}^{t}|C(t,\tau)|\,d\tau-\int_{t}^{\infty}|C(s,t)|\,ds\ge 0;$ 3) $a_0(t)\ge a_{00}>0,$ there is a function $a_0^*(t)\in L^1(J,R_+)$ such that $a_0'(t)\le a_0^*(t)a_0(t);$ 4) $A_i(t)>0, B_i(t)\ge 0,$ $B_i'(t)\le 0,$ there are functions such $A_i^*(t)\in L^1(J,R_+), c_i(t)$ that $A_i'(t)\le A_i^*(t)A(t),$ $(E_i^{(k)}(t))^2\le B_i^{(k)}(t)c_i^{(k)}(t)\ (i=1..n; k=0,1);$

4) $\int_{t_0}^{t}[|Q_{i\tau}'(t,\tau)|(A_i(\tau))^{-1/2}+|K_0(t,\tau)|]d\tau+|f_0(t)|\in L^1(J,R_+)\ (i=1..n).$

Then all solutions and their first derivatives of IDE (1) are bounded on *J* and for any solution *x(t)* of IDE (1) the following statements are true:

$$\Delta(t)(x'(t))^2\in L^1(J,R_+), \qquad (2)$$

$$A_i(t)(X_i(t,t_0))^2=O(1)\ (i=1..n), \qquad (3)$$

where

$$X_i(t,t_0)\equiv\int_{t_0}^{t}\psi_i(\eta)x'(\eta)d\eta\ (i=1..n).$$

To prove this theorem, we construct the following Lyapunov functional:

$$V(t;x)=(x'(t))^2+\int_{t_0}^{t}\Delta(s)(x'(s))^2ds+a_0(t)(x(t))^2+\int_{t_0}^{t}[\int_{t_0}^{\infty}|C(s,\tau)|ds](x(\tau))^2d\tau+$$

$$+\sum_{i=1}^{n}[P_i(t)(X_i(t,t_0))^2 - 2E_i(t)X_i(t,t_0)+c_i(t)]. \qquad (4)$$

Further, similarly to the proof of Theorem 1.14 [3, p. 74-76], it is taken $\frac{dV(t;x)}{dt}$ by virtue of IDE (1), and the idea of obtaining a differential comparison equation [4] is developed using the lemma on the differential inequality [5,6] and at the end the transition to the integral inequality to which Lemma 1 [7] is applied.

Based on relation (2), similarly to Corollary 3.5 [3, p. 117], we formulate

Corollary 1. If all conditions of the theorem are satisfied and $\Delta(t)\ge\Delta_0>0$ (respectively $\Delta(t)>0,(\Delta(t))^{-1}\in L^1(J,R_+\setminus\{0\})$), then for any solution *x (t)* to IDE (1) $x'(t)\in L^2(J,R)$ (respectively $L^1(J,R)$).

The second statement of Corollary 1 implies $|\lim_{t\to\infty}x(t)\}<\infty.$

From relation (3) we apply Lemma 3.3 [3, p. 111], we obtain

Corollary 2. If all the conditions theorem and $A_j(t)\ge A_{j0}>0,$ $\psi_j(t)>0,\psi_j'(t)\ge 0\ (1\le j\le n),$ any solution *x(t)* IDE (1) bounded on J .

Let's give a simple illustrative example.

Example. The IDE

$$x''(t)+5(\sqrt{t}+3)e^{\sqrt{t}}a_1(t)+e^{\frac{t+1}{t+2}}a_0(t)+$$
$$+\int_0^t[(t+1)(\tau+1)\sqrt{e^{-7t}(t-\tau)+4}-$$
$$-\frac{10}{(t+\tau+1)^3}-\frac{1}{(t+\tau+2)^{3/2}}]x'(\tau)d\tau=-t-1-\frac{3\sqrt[5]{\sin t}}{t^2+1},t\ge 0$$

satisfies all conditions of Theorem and Corollaries 1, 2, here

$$t_0=0,\Delta(t)\equiv 10(\sqrt{t}+3)e^{\sqrt{t}}-\frac{2}{\sqrt{t+2}},$$
$$a_{00}=\sqrt{e},a_0^*\equiv\frac{1}{(t+1)(t+2)},$$

$$n = 1, \psi_1(t) \equiv (t+1), P_1(t) \equiv 2, A_1(t) \equiv 1, B_1(t) \equiv 1, E_1(t) \equiv -1, c_1(t) \equiv 1,$$

$$K_0(t,\tau) \equiv -\frac{10}{(t+\tau+1)^3}, f_0(t) \equiv -\frac{3\sqrt[5]{\sin t}}{t^2+1}, C(t,\tau) \equiv -\frac{1}{(t+\tau+2)^{3/2}},$$

Hence, all solutions and their first derivatives of the reduced IDE are bounded on the semiaxis R_+, $|\lim_{t\to\infty} x(t)| < \infty, t \to \infty$ and for any solution *x (t) the* statement.

We will show that the task we set can be solved.

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MSC 15A83

CREATION OF THE DATABASE OF LOW-ORDER M-MATRICES IS AN IMPORTANT STEP OF THE DECOMPOSITION METHOD

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In this paper, an important stage of the decomposition method for creating a database of low-order M-matrices is considered. In this case, some properties of the arifmetic progression are used.
Keywords: decomposition method, magic square, square block matrices, square Durer, square constant, arifmetic proqression.

Бул эмгекте төмөнкү тартиптеги М-матрицалардын маалымат тартибин түзүү үчүн декомпозиция ыкмасынын маанилүү этабы каралат. Ага кошумча, арифметикалык прогрессиянын кээ бир касиеттери колдонулат.
Урунттуу сөздөр: декомпозиялоо методу, магиялык квадрат, Дюрердин квадраты, квадраттык блоктук матрицалар, квадраттын константасы, арифметикалык прогрессия.

В данной работе рассматривается важный этап метода декомпозиции создания базы данных М-матриц низкого порядка. При этом используется некоторые свойства арифметичекой прогрессии.
Ключевые слова: метод декомпозиции, магический квадрат, квадрат Дюрера, квадратные блочные матрицы, константа квадрата, арифмитическая прогрессия.

In [1], a method was proposed for constructing high-order *M*-matrices using the decomposition method. Due to the properties of the constants of squares and the preservation of symmetry by *M*-matrices, it is precisely the method of constructing the same symmetric matrices of an ever higher order that makes it possible. In this case, some properties of the arithmetic progression are used. Note that the set of values of the squares constants is modeled by a second-order difference equation of the form

$$u_{n+2} = 2u_{n+1} - u_n, \quad n \geq 1 \qquad (1)$$

with appropriate initial conditions. According to Euler's method, the general solution of equation (1) has the form

$$u(n) = c_1 + c_2 n$$

since the roots of the characteristic equation $\lambda^2 - 2\lambda + 1 = 0$: $\lambda_1 = \lambda_2 = 1$ coincide. Square constants forming an arithmetic progression act as operators.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Therefore, in this method of constructing high-order *M*-matrices, the starting point is the presence of lower-order *M*-matrices. Therefore, in this method, it becomes necessary to create a database of *M*-matrices of the lowest order. In addition, the authors have at their disposal a set of constructed *M*-matrices from the fourth to tenth order, which makes it possible to construct high-order *M*-matrices. In this article, we will illustrate the construction of an *M*-matrix of the 12th order on the basis of the available *M*-matrices of the third and fourth orders. Recall that the first chosen *M*-matrix of the fourth order is known as the "Durer square".

6	7	2
1	5	9
8	3	4

Fig. 1

13	2	12	7
16	3	9	6
1	14	8	11
4	15	5	10

Fig 2.

We will construct a 12th order *M*-matrix using a 3 * 4 decomposition method. Now we divide the set of numbers from 1 to 144 into 9 groups, each group contains 16 numbers in ascending order: I-th group - 1, 2, 3, ..., 16, II-th group - 17, 18, ..., 31, 32, and etc. , VIII th group - 113, 114, …, 127, 128, IX th group - 129, 130, …, 143, 144. Further, we will place each group of numbers according to fig. 2, counting the initial number as the shifted number 1, and the last number as the shifted number 16. For example, in the II th group I will place the number 17 instead of 1; the number 18 is placed instead of 2, etc., the last number of this group is 32 instead of 16. To calculate the constant of the squares of each group, we proposed the formula

$$S = \frac{n^2 + 1}{2} n + kn$$

k - shift amount: for the second group it is equal to $k = 16$, for the third group - $k = 32$, etc. For our case $n = 4$. Then the constants of the squares of the created groups form an arithmetic progression with the difference $d = 16 \cdot 4 = 64$. We have the first 9 members of the arithmetic progression: $a_1 = 34$, $a_2 = 98$, $a_3 = 162$, ... , $a_9 = 546$ and they will be placed in full compliance with the numbers in Fig. 1, i.e.

$$\begin{pmatrix} a_6 & a_7 & a_2 \\ a_1 & a_5 & a_9 \\ a_8 & a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 354 & 418 & 98 \\ 34 & 290 & 546 \\ 482 & 162 & 226 \end{pmatrix}$$

Fig. 3

Note that the constant of the squares of the last *M*-matrix is 870.

Further, we will consider the obtained 9 square constants as expanding order operators. An *M*-matrix of the 12th order is considered as a nested 3 * 4, or rather a block matrix. We consider each element of the third-order *M*-matrix as a block and expand it, keeping its location in Fig. 3. So, for example, the constant 34 will consist of *M*-matrices of the fourth order of the form

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Due to the constants of the square, each block can be painted in an arbitrary way. We have the right to take any other M-matrices of the fourth order from the database, for example

13	2	12	7
16	3	9	6
1	14	8	11
4	15	5	10

Fig. 4

For the constant 98 and 162, we can take the *M*-matrix of the fourth order built on the basis of the last Fig. 4 contained in our database:

29	18	28	23
32	19	25	22
17	30	24	27
20	31	21	26

45	34	44	39
48	35	41	38
33	30	40	43
36	47	37	42

We will continue this procedure for all square constants in Fig. 3. As a result, instead of each constant of a square, we write down the *M*-matrix of our group, we get *M* - a matrix of the 12th order. Here is a fragment of the arrangement instead of constants 34, 98, 162 M-matrices of the 4th order of these groups.

								29	18	28	23
								32	19	25	22
								17	30	24	27
								20	31	21	26
13	2	12	7								
16	3	9	6								
1	14	8	11								
4	15	5	10								
				45	34	44	39				
				48	35	41	38				
				33	30	40	43				
				36	47	37	42				

Finally, we get one form of the *M*-matrix of the 12th order:

93	82	92	87	109	98	108	103	29	18	28	23
96	83	89	86	112	99	105	102	32	19	25	22
81	94	88	91	97	110	104	107	17	30	24	27
84	95	85	90	100	111	101	106	20	31	21	26
13	2	12	7	77	66	76	71	141	130	140	135
16	3	9	6	80	67	73	70	144	131	137	134
1	14	8	11	65	78	72	75	129	142	136	139
4	15	5	10	68	79	69	74	132	143	133	138
125	114	124	119	45	34	44	39	61	50	60	55
128	115	121	118	48	35	41	38	64	51	57	54
113	126	120	123	33	30	40	43	49	62	56	59
116	127	117	122	36	47	37	42	52	63	53	58

It's clear that

$$S = \frac{a_1 + a_{n^2}}{2} n = \frac{34 + 546}{2} 3 = 870$$

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MSC 34K20, 45J05

APPENDIX OF THE "PREDATOR - VICTIM" MATHEMATICAL MODEL TO THE DYNAMICS OF HEALTH LEVEL ISSYK KUL

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The mathematical model of the interaction of evaporation and runoff of closed water bodies is written by the "predator - prey" system and it is shown that the phase trajectories form a rest point of the "center" type.

Key words: systems "predator - prey", the volume of runoff and the volume of evaporation of the reservoir, phase trajectories, rest point of the "center" type.

Туюк суу объектилеринин бууланусу менен кирген суунун өз ара аракеттенүүсүнүн математикалык модели «жырткыч – олжо» системасы аркылуу жазылган жана фазалык траекториялар «борбор» тибиндеги «тынчтык» чекити болоору көрсөтүлгөн.

Урунттуу сөздөр: “жырткыч – олжо” системалары, агын суулардын көлөмү жана суу сактагычтын буулануу көлөмү, фазалык траекториялар “борбор” тибиндеги «тынчтык» чекити.

Математическая модель взаимодействия испарения и стока замкнутых водоемов записана системой «хищник - жертва» и показано, что фазовые траектории образуют точку покоя типа «центр».

Ключевые слова: системы «хищник - жертва», объем стока и объем испарения водоема, фазовые траектории, точку покоя типа «центр».

In the qualitative theory of differential equations [1], it is known that the phase trajectories of the "predator - prey" system form a rest point of the "center" type.

The basis for compiling a model of the constituent elements of the water balance are the following provisions [2]. The first of them is an active factor, which is characterized by variability of hydrological and climatic conditions, i.e. inflow of water and intensity of visible evaporation. The second is a reactive factor that seeks to smooth out level fluctuations.

A mathematical model of the interaction of evaporation and runoff of closed water bodies can be written by the Lotka-Volterra system of equations:

$$\begin{cases} \dot{x}(t) = \alpha x(t) + \beta x(t) y(t), \\ \dot{y}(t) = \gamma\, y(t) + \delta\, x(t) y(t), \end{cases}$$

where $x(t)$ and $y(t)$ are the volume of runoff and evaporation of the reservoir at time t, α is the runoff rate $\alpha > 0, \beta, \delta$ characterizes the interaction of the active and reactive factors $\beta < 0, \delta > 0, \gamma$ - characterizes that in the absence of runoff, the volume of visible evaporation decreases, i.e. $\gamma < 0$.

For a complete statement of the problem of the dynamics of the components of closed reservoirs, it is necessary to set the considered time interval $t \in [t_0, T]$, the volume of runoff $x(t_0) = x_0$ and the volume of evaporation $y(t_0) = y_0$ at the initial moment of time. It is clear that the autonomous system (2) has a rest point $\left(-\frac{\gamma}{\delta}, -\frac{\alpha}{\beta}\right)$, which is obtained by equating the right-hand side of system (2) to zero. It is clear that the resting point is in the first quadrant of the $x0y$ phase plane. The matrix of the linearized system (2) in the vicinity of the rest point $\left(-\frac{\gamma}{\delta}, -\frac{\alpha}{\beta}\right)$ has eigenvalues $\pm i\sqrt{\alpha\gamma}$. Since the real part of the characteristic numbers is equal to zero, Lyapunov's theorem on stability in the first approximation is not applicable. Therefore, integrating the equation with separable variables

$$\frac{dy}{dx} = \frac{\gamma y + \delta xy}{\alpha x + \delta xy}.$$

Find the integral (implicit solution) V of system (2) in the domain

$$x > 0,\ y > 0 : V(x, y) = \delta x - \beta y + \gamma \ln x - \alpha \ln y.$$

It can be verified that the curves defined by the equation $V(x, y) = C$ are closed curves surrounding the rest points and filling the entire region $x > 0, y > 0$ (fig. 1).

Indeed, let (x, y) be a solution to system (2). The function $V(x(t), y(t))$ is continuously differentiable along the trajectory of this solution. Let us calculate its total derivative $\frac{dV}{dt}$ with respect to time, compiled by virtue of system (2):

$$\frac{d}{dt}V(x(t), y(t)) = \delta\dot{x} - \beta\dot{y} + \frac{\gamma}{x}\dot{x} - \frac{\alpha}{y}\dot{y} = \left(\delta + \frac{\gamma}{x}\right)\dot{x} - \left(\beta + \frac{\alpha}{y}\right)\dot{y} =$$
$$= \left(\delta + \frac{\gamma}{x}\right)(\alpha x + \beta yx) - \left(\beta + \frac{\alpha}{y}\right)(\gamma y - \delta xy) =$$
$$= \alpha\delta x + \alpha y + \delta\beta yx + \gamma\beta y - [\beta\gamma y + 2\gamma + \beta\delta yx + 2\delta x] = 0.$$

Since these curves are the trajectories of system (2), the rest point $\left(-\frac{\gamma}{y}, \frac{\alpha}{\delta}\right)$ - center. Summing up, we formulate the following result.

Fair

Periodic cycle theorem. Fluctuations in runoff and visible evaporation in a closed hydra ecological system are periodic.

Comment. Lake Issyk-Kul is a typical representative of a closed body of water. If we neglect the moisture carried by air masses outside the Issyk-Kul basin and the influence of the temperatures of the environment surrounding the basin, the basin can be considered a closed hydroecological system. Under these assumptions, the periodic cycle theorem will be valid for the Issyk-Kul basin, from which it follows that the lake level fluctuations are periodic.

An illustrative example of the interaction of runoff and apparent evaporation in a closed hydroecological system is given below.

The significance of the periodic cycle theorem can be estimated from the fact that one of the main problems of mathematical ecology is the problem of ecosystem sustainability.

Consider the Volterra system

$$\begin{cases} \dot{x} = (\alpha - \beta y)x, \\ \dot{y} = (-\gamma + \delta x)y, \end{cases} \quad (3)$$

where $\alpha, \beta, \gamma, \delta > 0.$

With this model, we will describe the interaction of runoff and visible evaporation in a closed hydrological system.

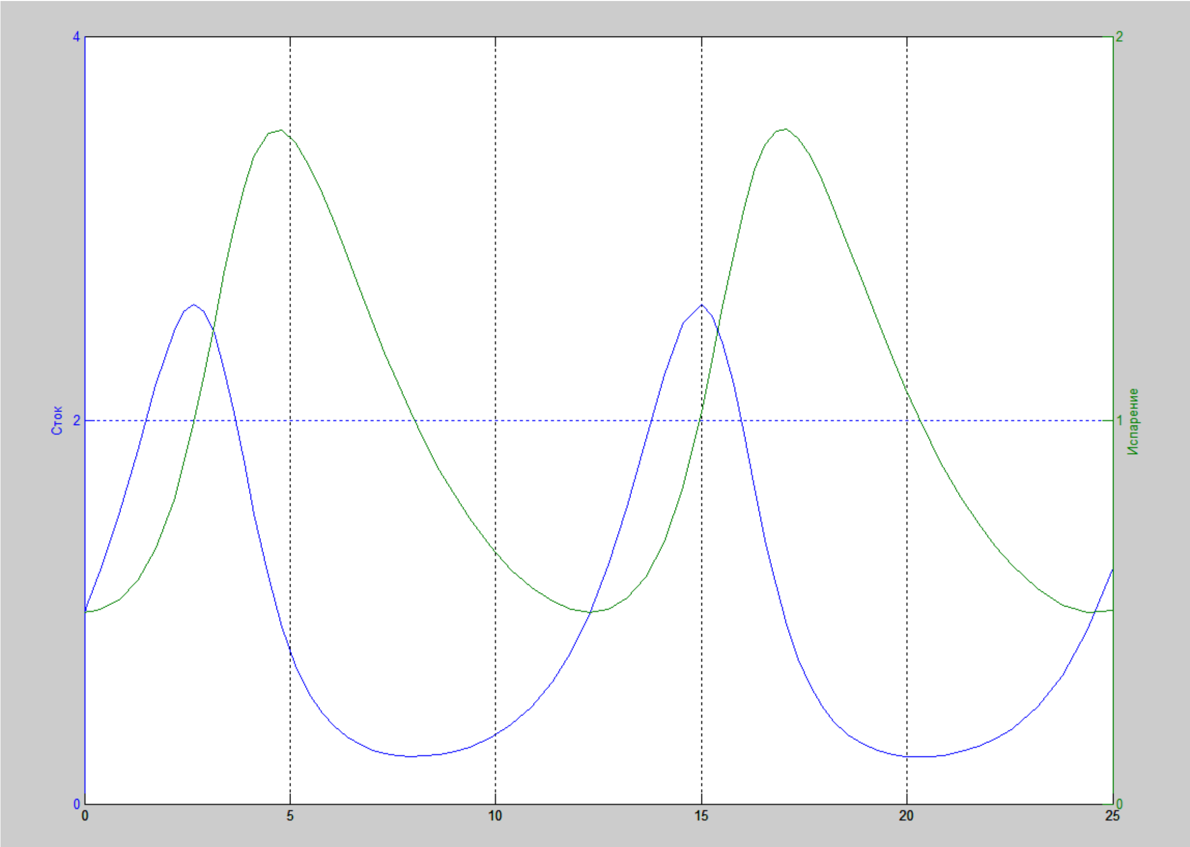
Consider the phase portrait of the Volterra system for

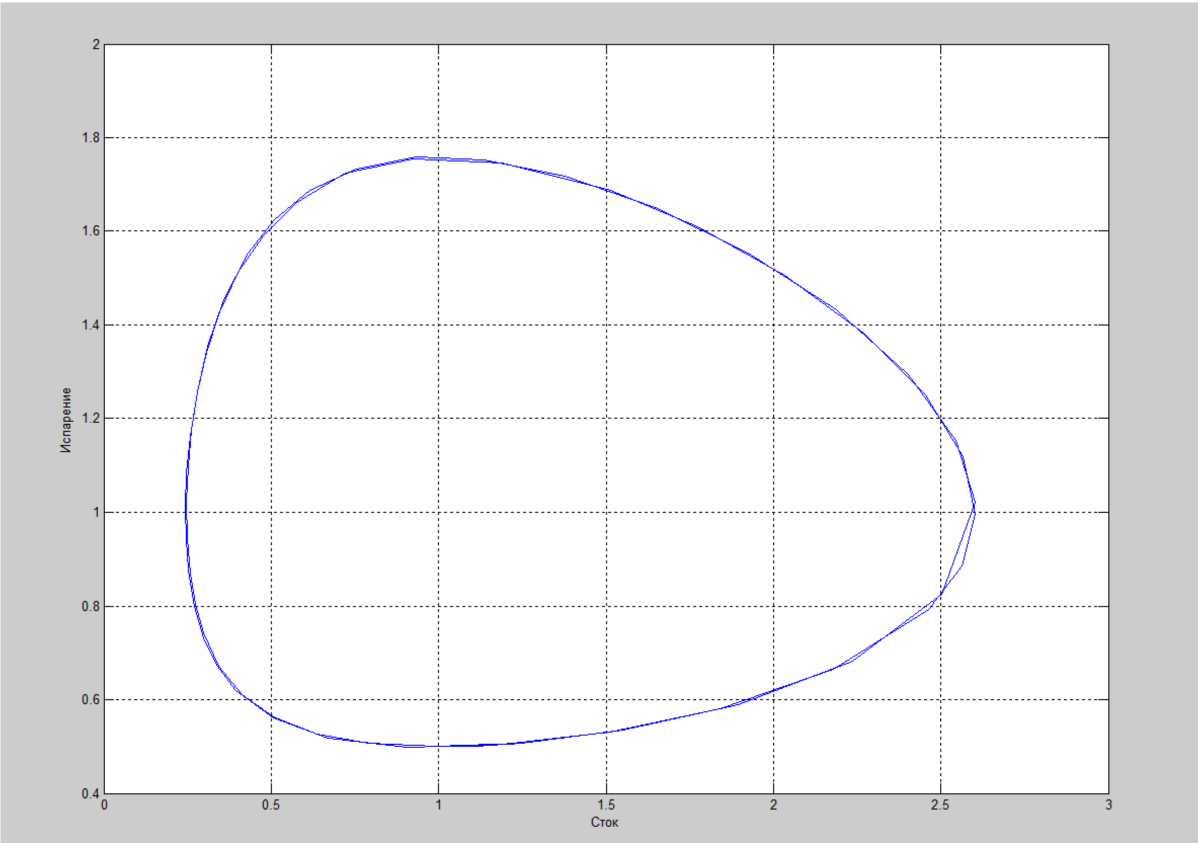
$$\alpha = 1, \ \beta = 1, \ \gamma = 0,3, \ \delta = 0,3,$$

i.e.

$$\begin{cases} \dot{x} = (1 - y(t))x(t), \\ \dot{y} = 0,3(-1 + x(t))y(t). \end{cases}$$

Let's build graphs of its solutions with the initial condition $x(t) = 1, \ \ y(t) = 0,5.$ With the program listed in Appendix 16:





It is seen that the process has an oscillatory character. Applying 2:1 for a given initial ratio of runoff and evaporation volumes, both values first increase. When the evaporation volume reaches 2.5, the flow volume does not have time to recover and the flow begins to decrease. A decrease in the amount of water runoff after a while begins to affect visible evaporation, and when the runoff reaches the value $x = \frac{\gamma}{\delta} = 1,0$ (at this point $\dot{y} = 0$), the evaporation volume also begins to decrease along with a decrease in the runoff volume. A decrease in runoff and evaporation occurs until the evaporation volume reaches the value $y = \frac{\alpha}{\beta} = 1$ (at this point $\dot{x} = 0$). From this moment, the volume of runoff begins to grow, after a while, with an increase in runoff, the area of the lake mirror will also increase, which will provide an increase in evaporation, both values increase and the process repeats over and over again. The graph clearly shows the periodic nature of the process. The flow volume fluctuates around the value $x = 1,\ y = 1$.

The periodicity of the process is clearly visible on the phase plane - the phase curve ($x(t),\ y(t)$ is a closed line, the leftmost point of this curve is the point at which the runoff volume reaches the smallest value. The rightmost point $x = 2,5,\ \ y = 1$ is the peak point of the runoff volume. Between these points, the evaporation volume first decreases, to the lower point of the phase curve $y = 0,5$, where it reaches the smallest value, and then grows to the upper point of the phase curve ($x = 1,\ y = 1,7$). The phase curve covers the point $x = 1,\ y = 1.$, i.e. stationary state $\dot{x} = 0,\ \dot{y} = 0$. If at the initial moment the system was at the stationary point $x = 1,\ \ y = 1$, then the solutions $(x(t),\ y(t))$ will not change in time, will remain constant. Any other initial state leads to periodic fluctuations of solutions. The non-elliptic shape of the trajectory covering the center reflects the non-harmonic nature of the oscillations.

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MSC 45A05, 45B05

APPLICATION OF THE SUMMARY-DIFFERENCE METHOD WITH A REGULARIZER TO CONSTRUCT AN ASYMPTOTIC SOLUTION TO THE BOUNDARY VALUE PROBLEM OF A SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

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Finite-difference equations are a convenient mathematical model for describing the problem of mathematical physics, financial analysis and other problems of science and technology. Difference equations arise in the numerical solution of various classes of differential equations and integro-differential equations as with a small parameter, with significant nonlinearities.

In this paper, we consider methods for constructing an asymptotic solution to a boundary value problem for a system of nonlinear difference equations, with a small parameter, in the case when the boundary value problem cannot be related to the Cauchy problem.

Key words: Sum-difference method, boundary value problem, small parameter, regularizer, finite-difference equation.

Чектүү –айырмадагы теңдемелер математикалык физика, финансы анализдин жана бащка илимдерде, техникада кездешүүчү маселелердин математикалык моделдерин сүрөттөп, аны изилдөөнүн натыйжалуу ыкмасы болуп эсептелет.

Бул макалада чектүү айырмадагы, кичине параметрди кармаган теңдеме үчүн чектик маселенин, асимптотикалык чыгарылышын табуунун, чектик маселе Кошинин маселеге келтирилбей турган учурдагы, ыкманы иштеп чыгуу маселеси каралат.

Уруттуу сөздөр: суммалап-айырмалоо ыкма, чектик маселе, кичине параметр, регуляризатор, чектүү айырмадагы теңдеме.

Конечно-разностные уравнения являются удобной математической моделью при описании задачи математической физики, финансового анализа и другие задачи науки, техники. Разностные уравнения возникают при численном решении различных классов дифференциальных уравнений и интегро-дифференциальных уравнений как с малым параметром, так и с существенными нелинейностями.

В данной работе рассматривается способы построения асимптотического решения краевой задачи системы нелинейной разностного уравнения с малым параметром, в случае, когда краевая задача не может быть сведено к задаче Коши.

Ключевые слова: Суммарно-разностный метод, краевая задача, малый параметр, регуляризатор, конечно-разностные уравнение.

Consider the boundary value problem

$$x(k+1,\varepsilon) = x(k,\varepsilon) + g(k) + \varepsilon f(k,\varepsilon), \qquad (1)$$

$$Ax(0,\varepsilon) + Bx(N,\varepsilon) = d, \qquad (2)$$

where x, f, d are vectors, A, B are constant matrices, the function is a sufficiently smooth function with respect to the variables k, x.

Currently, there are various approaches to the study of asymptotic solutions, finite-difference equations with a small parameter. These methods include the method of small parameter A. Poincaré [2], the method of elimination [1], the method of auxiliary equations [1], etc.

In [3; 4], a method of integro-differential equations for studying boundary value problems of a system of linear differential equations, a continuous analogue of the sum-difference method, was developed. The ideas of the integro-differential equation method are extended to a boundary value problem of the form (1), (2) in this article.

If det$B \neq 0$, then the problem of studying the boundary value problem can be reduced to the Cauchy problem of the form

$$x(k+1,\varepsilon)-x{=}g(k)+\varepsilon f(k,x)-\frac{1}{N}\sum_{k=0}^{N-1}(g(k)\ +\varepsilon f(k,x(k,\varepsilon))+$$

$$+\frac{1}{N}[B^{-1}d-(B^{-1}A+E)]x(0,\varepsilon),$$

$$x(0,\varepsilon)=a_0+\varepsilon a_1+\varepsilon^2 a_2+\cdots \quad .$$

Consider the case det$B$ = 0. In this case, the boundary value problem (1), (2) cannot be reduced to the Cauchy problem, because in this case the inverse matrix B^{-1} does not exist. To work around this problem together matrices. Consider a regularized matrix $B_0(\tau)$ such that $B_0(\tau) \to B$ for $\tau \to 0$, for which det $B_0(\tau)$ and consider the boundary value problem

$$x(k+1,\varepsilon,\tau)=x(k,\varepsilon,\tau)+g(k)+\varepsilon f(k,x(k,\varepsilon,\tau)), \qquad (3)$$

$$Ax(0,\varepsilon,\tau)+B_0\,x(N,\varepsilon,\tau)=d. \qquad (4)$$

Lemma. If det$B_0(\tau) \neq 0$ for $\tau \to 0$, then the boundary value problem (3), (4) can be reduced to the Cauchy problem

$$x(k+1,\varepsilon,\tau)-x(k,\varepsilon,\tau)=g(k)+\varepsilon f(k,x(k,\varepsilon,\tau))-\frac{1}{N}\sum_{k=0}^{N-1}[g(k)+$$

$$+\varepsilon f(k,x(k,\varepsilon,\tau))] + \frac{1}{N}[B_0^{-1}(\tau)d - (B_0^{-1}(x)A + E)x(0,\varepsilon,\tau)], \qquad (5)$$

$$x(0,\varepsilon,\tau) = a_0(\tau) + \varepsilon a_1(\tau) + \varepsilon^2 a_2(\tau) + \cdots, \qquad (6)$$

where $a_0(\tau)$, $a_1(\tau)$,..., subject to the choice of numbers.

Proof. Adding the parameter α to the right-hand side, we write system (3) in the form

$$x(k+1,\varepsilon,\tau) = x(k,\varepsilon,\tau) + g(k) + \varepsilon f(k,x(k,\varepsilon,\tau)) + \alpha, \qquad (7)$$

and sequentially supplying k = 0, 1, 2, ... in (7) we get

$$x(k,\varepsilon,\tau) = x(0,\varepsilon,\tau) + \sum_{i=0}^{N-1}[g(i) + \varepsilon f(i,x\ \ i,\varepsilon,\tau\ \ )] + k\alpha,$$

for $k = N$ we have

$$x(N,\varepsilon,\tau) = x(0,\varepsilon,\tau) + \sum_{i=0}^{N-1}[g(i) + \varepsilon f(i,x(i,\varepsilon,\tau))] + N\alpha. \qquad (8)$$

Substituting (8) into (4), we have

$$\mathrm{A}x(0,\varepsilon,\tau) + \mathrm{B}_0(\tau)[x(0,\varepsilon,\tau) + \sum_{i=0}^{N-1}[g(i) + \varepsilon f(i,x(i,\varepsilon,\tau))] + N\alpha = d.$$

From here we find

$$\alpha(\tau) = \frac{1}{N}[B_0^{-1}(\tau)d - (B_0^{-1}(\tau)A + E)x(0,\varepsilon,\tau) - \frac{1}{N}\sum_{k=0}^{N-1} f(k,x(k,\varepsilon,\tau))],$$

and from (7) we obtain

$$x(k+1,\varepsilon,\tau) - x(k,\varepsilon,\tau) = g(k) + \varepsilon f(k,x(k,\varepsilon,\tau)) -$$

$$-\frac{1}{N}\sum_{i=0}^{N-1}(g(k) + \varepsilon f(k,x(k,\varepsilon,\tau)) + \frac{1}{N}[(B_0^{-1}(\tau)d - B_0^{-1}(\tau)A + E)x(0,\varepsilon,\tau)],$$

$$x(0,\varepsilon,\tau) = a_0(\tau) + \varepsilon a_1(\tau) + \varepsilon^2 a_2(\tau) + \cdots.$$

The lemma is proved.

We find the asymptotic solution of problem (5), (6) in the form

$$x(k,\varepsilon,\tau) = x_0(k,\tau) + \varepsilon x_1 + \varepsilon^2 x_2(k,\tau) + \cdots. \qquad (9)$$

Putting (9) into (5) we get

$$x_0(k+1,\tau) + \varepsilon x_1(k+1,\tau) + \varepsilon^2 x_2(k+1,\tau) + \cdots - x_0(k,\tau) - \varepsilon x_1(k,\tau) -$$

$$-\varepsilon^2 x_2(k,\tau) - \cdots = g(k) + \varepsilon f(k, x_0(k,\tau) + \varepsilon x_1(k,\tau) + \cdots) -$$

$$-\frac{1}{N}\sum_{k=0}^{N-1}[g(k) + \varepsilon f(k_0, x_0(k,\tau)) + \varepsilon x_1(k,\tau) + \cdots)] + \frac{1}{N}[B_0^{-1} -$$

$$-(B_0^{-1}(x)A + E)(a_0(\tau) + \varepsilon a_1(\tau) + \varepsilon^2 a_2(\tau) + \cdots)],$$

$$x(0,\varepsilon,\tau) = a_0(\tau) + \varepsilon a_1(\tau) + \varepsilon^2 a_2(\tau) + \cdots \ .$$

From here we get

$$\varepsilon^0: \ x_0(k+1,\tau) - x_0(k,\tau) = g(k) - \frac{1}{N}\sum_{k=0}^{N-1} g(k) + \frac{1}{N}[B_0^{-1}(\tau)d -$$

$$-(B_0^{-1}(x)A + E)(a_0(\tau))], \qquad (10_0)$$

$$x_0(0,\tau) = a_0(\tau). \qquad (11_0)$$

$$\varepsilon^1: \ x_1(k+1,\tau) - x_1(k,\tau) = f(k, x_0(k,\tau)) - \frac{1}{N}\sum_{k=0}^{N-1} f(k, x_0(k,\tau)) -$$

$$-(B_0^{-1}(x)A + E)a_0(\tau), \qquad (10_1)$$

$$x_1(0,\tau) = a_1(\tau). \qquad (11_1)$$

$$\varepsilon^2: \ x_2(k+1,\tau) - x_2(k,\tau) = f_x(k, x_0(k,\tau))x_1(k,\tau) -$$

$$-\frac{1}{N}\sum_{k=0}^{N-1} f_k(k, x_0(k,\tau))x_1(k,\tau) - (B_0^{-1}(\tau)A + E)a_2(\tau), \qquad (10_2)$$

$$x_2(0,\tau) = a_2(\tau). \qquad (11_2)$$

… … … … … … … … … … …

$$\varepsilon^i: \ x_i(k+1,\tau) - x_i(k,\tau) = f_x(k, x_0(k,\tau))x_{i-1}(k,\tau) -$$

$$-\frac{1}{N}\sum_{k=0}^{N-1} f_k(k, x_0(k,\tau))x_{i-1}(k,\tau) - (B_0^{-1}(\tau)A + E)a_i(\tau), \qquad (10_i)$$

$$x_i(0,\tau) = a_i(\tau). \qquad (11_i)$$

… … … … … … … … … … …

We choose $a_0(\tau)$ from (10_0) so that

$$-\frac{1}{N}\sum_{k=0}^{N-1} g(k) + \frac{1}{N}[B_0^{-1}(\tau)d - (B_0^{-1}(\tau)A + E)a_0(\tau)] = 0.$$

From here we find

$$a_0(\tau) = (B_0^{-1}(\tau)A + E)^{-1}(B^{-1}d - \sum_{k=0}^{N-1} g(k)). \qquad (12)$$

Thus, the expansion term $x_0(k,\tau)$ is a solution to the Cauchy problem of the form

$$x_0(k+1,\tau) = x_0(k,\tau) + g(k),$$

$$x_0(0,\tau) = (B^{-1}(\tau)A + E)^{-1}(B^{-1}d - \sum_{k=0}^{N-1} g(k)).$$

Solving this problem, we get

$$x_0(k,\tau) = (B_0^{-1}(\tau)A + E)^{-1}(B^{-1}d - \sum_{k=0}^{N-1} g(k) + \sum_{i=0}^{k-1} g(i)). \qquad (13)$$

Similarly, from (10_1), (11_1)-(10_i), (11_i) we find the expansion terms $x_1(k,\tau), x_2(k,\tau), \ldots, x_i(k,\tau), \ldots$ in the form

$$x_1(k,\tau) = -(B_0^{-1}(\tau)A + E)^{-1}\sum_{k=0}^{N-1} f(k, x_0(k,\tau)) + \sum_{i=0}^{k-1} f(i, x_0(i,\tau)),$$

$$x_2(k,\tau) = -(B_0^{-1}(\tau)A + E)^{-1}\sum_{k=0}^{N-1} f_x(k, x_0(k,\tau)x_1(k,\tau) + \sum_{i=0}^{k-1} f_x(i, x_0(i,\tau)x_1(k,\tau),$$

… … … … … … … … … … …

$$x_j(k,\tau) = -(B_0^{-1}(\tau)A + E)^{-1}\sum_{k=0}^{N-1} f_x(k, x_0(k,\tau)x_{j-1}(k,\tau) + \sum_{i=0}^{k-1} f_x(k, x_0(i,\tau)x_{j-1}(k,\tau)$$

$k = 1, 2, \ldots, N$.

Theorem. If, in the domain $(k, x) \in D \times D_1$, where $D \in R_1 = (-\infty, \infty)$, $D_1 \subset E_n$, the condition $|f_x(t,x)| \le K$, K is a constant positive matrix, then

$$|\, x(k,\varepsilon,\tau) - X_n(k,\varepsilon,\tau)| \le (E - \varepsilon(N-1)K)^{-1}\varepsilon^{n+1}.$$

Proof. Let $x = x(k,\varepsilon,\tau)$ be an asymptotic solution to the boundary value problem (3), (4) $X_n(k,\varepsilon,\tau)$ is a partial sum.

We put

$$u(k,\varepsilon,\tau) = x(k,\varepsilon,\tau) - X_n(k,\varepsilon,\tau),$$

then for the difference $u(k,\varepsilon,\tau)$ we obtain the Cauchy problem of the form

$$u(k+1,\varepsilon,\tau) - u(k,\varepsilon,\tau) = \varepsilon f(k, u(k,\varepsilon,\tau) + X_n(k,\varepsilon,\tau)) + g(k) -$$
$$-X_n(k+1,\varepsilon,\tau) - X_n(k,\varepsilon,\tau)),$$

$$u(0,\varepsilon,\tau) = O(\varepsilon^{n+1}).$$

Hence we have

$$u(k+1,\varepsilon,\tau) - u(k,\varepsilon,\tau) = \varepsilon[f(k,u(k,\varepsilon,\tau) + \mathrm{X}_n(k,\varepsilon,\tau) - f(i,\mathrm{X}_n\ \ i,\varepsilon,\tau\ )]+$$
$$+g(k) + \varepsilon f(k,\mathrm{X}_n(k,\varepsilon,\tau)) - (\mathrm{X}_n(k+1,\varepsilon,\tau) - \mathrm{X}_n(k,\varepsilon,\tau)) \qquad (14)$$

$$|u(k,\varepsilon,\tau)| \le |u(0,\varepsilon,\tau)| +$$
$$+\varepsilon\sum_{i=0}^{k-1}\left|f(i,u(i,\varepsilon,\tau)+X_n(i,\varepsilon,\tau)) - f(i,X_n(i,\varepsilon,\tau)-X_n(i,\varepsilon,\tau))\right| +$$
$$+\sum_{i=0}^{k-1}\left|g(i)+\varepsilon f(i,X_n(i,\varepsilon,\tau)) - (X_n(i+1,\varepsilon,\tau)-X_n(i,\varepsilon,\tau))\right| \le$$
$$\le Q(\varepsilon^{n+1}) + \varepsilon\sum_{i=0}^{k-1}\int_0^1\left|f_x(i,x_n(i,\varepsilon,\tau)+(1+\theta)u(i,\varepsilon,\tau))u(i,\varepsilon,\tau)\right|d\theta + O(\varepsilon^{n+1}) \le$$
$$\le Q(\varepsilon^{n+1}) + \varepsilon(N-1)K|u(k,\varepsilon,\tau)|.$$

From here

$$|\, u(k,\varepsilon,\tau)| \le (E - \varepsilon(N-1)K)^{-1}Q(\varepsilon^{n-1}).$$

In this way

$$|\, x(k,\varepsilon,\tau) - \mathrm{X}_n(k,\varepsilon,\tau)| \le (E - \varepsilon(N-1)K)^{-1}\varepsilon^{n-1}.$$

The theorem is proved.

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MSC 45A05, 45B05

ON THE UNIQUENESS OF SOLUTIONS OF VOLTERRA LINEAR INTEGRAL EQUATIONS OF THE FIRST KIND ON THE SEMI-AXIS

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In the present article the theorem about uniqueness of Volterra linear integral equations of the first kind on the semi axis, with method of nonnegative quadratic forms and functional analysis methods.

Key words: Volterra linear inteqral equations, first kind, the semiaxis, uniqueness of solutions.

Бул макалада терс эмес квадраттык формалар усулунун, функционалдык анализдин усулдарынын жардамы менен жарым октогу Вольтерранын биринчи түрдөгү сызыктуу интегралдык тендемелеринин чечимдеринин жалгыздыгы далилденди.

Урунттуу сөздөр: Вольтерранын сызыктуу интегралдык тендемелери, биринчи түрдөгү, жарым ок, чечимдеринин жалгыздыгы.

В настоящей статье доказана теорема о единственности линейных интегральных уравнений Вольтерра первого рода на полуоси с использованием метода неотрицательных квадратичных форм и методов функционального анализа.

Ключевые слова: линейные интегральные уравнения Вольтерра, первый род, полуось, единственность решений.

1. Introduction

Many problems of the theory of integral equations of the first kind were studied in [1-15]. But fundamental results for Fredholm integral equations of the first kind were obtained in [11-12], where regularizing operators in the sence of M.M. Lavrent'ev were constructed. Rezults on non-classical Volterra integral equations of the first kind can be found in [1]. In [2, 6], problems of regularization, uniqueness and existence of solutions for Volterra integral and operator equations of the first kind are studied. In [13], for linear Volterra integral equations of the first and the third kind with smooth kernel, the existence of multiparameter family of solutions was proved. In [4, 5], on the basis of theory of Volterra integral equations of the first kind, various inverse problems were studied. In [8],uniqueness theorems were proved and reqularizing operators in the sense of Lavrent'ev were constructed for systems of linear Fredholm integral equations of the third kind. In [10], problems of uniqueness and stability of solutions for linear integral equations of the first kind with two independent variables were investigated. In [3, 9], based on a new approach, the existence and uniquenes of solutions of Fredholm integral equations and the system linear Fredholm integral equations of the third kinds were studied. In [15], uniqueness theorems were proved for the linear Fredholm integral equations of the first kind in the axis.

In the prezent paper, on the basis of the method of integral transformation, uniqueness theorems for the new class of linear Volterra integral equations of the first kind in the semiaxis were proved.

2. The linear Volterra integral equations of the first kind

Consider the linear Volterra integral equations of the first kind

$$Ku \equiv \int_{-\infty}^{t} H(t,s)u(s)ds = f(t), \quad t \in (-\infty, a] \tag{1}$$

where the $u(t)$ is the desired function on $(-\infty, a]$, the given function $f(t)$ is the continuous on $(-\infty, a]$,

$$\int_{-\infty}^{a}\int_{-\infty}^{t} |H(t,s)|^2 dsdt < \infty,$$

the given function $H(t,s)$ is continuous on the domain

$$G = \{(t,s) : -\infty < s \le t \le a\}.$$

Let C$(-\infty, a]$ denote the space of all functions continuous on $(-\infty, a]$. Here C(G) denote the space of all functions continuous on G.

Assume that the following conditions are satisfied:

$(i) H(t,s), H'_t(t,s), H'_s(t,s), H''_{ts}(t,s) \in C(G), \alpha(t) = \lim_{s\to-\infty} H(t,s), t \in (-\infty, a],$

$\alpha(t) \in C(-\infty, a], \alpha'(t) \le 0$ *for all* $t \in (-\infty, a], H''_{st}(t,s) \le 0$ *for all* $(t,s) \in G,$

$\alpha'(t) \in L_1(-\infty, a], \beta(s) = H'_s(a,s) \ge 0$ *for all* $s \in (-\infty, a], \beta(s) \in C(-\infty, a] \cap L_1(-\infty, a];$

$(ii) \sup_{(t,s)\in G} |H(t,s)| \le l < \infty, \ \sup_{t\in(-\infty,a]} \int_{-\infty}^{t} |H'_s(t,s)| ds \le l_1 < \infty,$

$H''_{ts}(t,s) \in L_1(G), \ \sup_{t\in[s,a]} \left| H'_s(t,s) \right| \le \gamma(s) \in L_1(-\infty, a];$

(iii) At least one of the following three conditions holds:

1) $\alpha'(t) < 0$ *for almost* $t \in (-\infty, a]$; 2) $\beta(s) > 0$ *for almost all* $s \in (-\infty, a]$;

3) $H''_{ts}(t,s) < 0$ *for almost all* $(t,s) \in G.$

Theorem. Let conditions (i), (ii) and (iii) be satisfied. Then the solution of the integral equation (1) is unique in $L_1(-\infty, a]$.

Proof. Let $u(t) \in L_1(-\infty, a]$ be a solution of the integral equation (1).

Multiplying both sides of the equation (1) by u(t) and integrating over the domain $(-\infty, a]$, we obtain.

$$\int_{-\infty}^{a} \int_{-\infty}^{t} H(t,s)u(s)ds\, u(t)dt = \int_{-\infty}^{a} f(t)u(t)dt. \qquad (2)$$

We shall introduce the notation

$$z(t,s) = \int_{s}^{t} u(\nu)d\nu, (t,s) \in G. \qquad (3)$$

Then from (3), we obtain

$$d_s z(t,s) = -u(s)ds, z(t,s)u(t)dt = \frac{1}{2} d_t(z^2(t,s)). \qquad (4)$$

Let us transform the integral on the left hand of the identity (2). Taking into account (3),(4) and integrating by parts, we have

$$\int_{-\infty}^{a} \int_{-\infty}^{t} H(t,s)u(s)u(t)dsdt = \int_{-\infty}^{a} \alpha(t)z(t,-\infty)u(t)dt + \int_{-\infty}^{a} \int_{-\infty}^{t} H'_s(t,s)z(t,s)u(t)dsdt.$$

Hence, applying Dirichlet’s formula, we obtain

$$\int_{-\infty}^{a} \int_{-\infty}^{t} H(t,s)u(s)u(t)dsdt = \frac{1}{2}\int_{-\infty}^{a} \alpha(t)d_t(z^2(t,-\infty)) + \frac{1}{2}\int_{-\infty}^{a} \left[\int_{s}^{a} H'_s(t,s) d_t(z^2(t,s)) \right] ds =$$

$$= \frac{1}{2}\alpha(a)z^2(a,-\infty) - \frac{1}{2}\int_{-\infty}^{a} \alpha'(t)z^2(t,-\infty)dt + \frac{1}{2}\int_{-\infty}^{a} \beta(s)z^2(a,s)ds - \qquad (5)$$

$$-\frac{1}{2}\int_{-\infty}^{a} \int_{s}^{a} H''_{ts}(t,s)z^2(t,s)dtds.$$

Taking into account (2) and applying Dirichlet’s formula from (5), we have

$$\frac{1}{2}\alpha(a)z^2(a,-\infty)-\frac{1}{2}\int\limits_{-\infty}^{a}\alpha^{'}(t)z^2(t,-\infty)dt+\frac{1}{2}\int\limits_{-\infty}^{a}\beta(s)z^2(a,s)ds-$$
$$-\frac{1}{2}\int\limits_{-\infty}^{a}\int\limits_{-\infty}^{t}H_{ts}^{''}(t,s)z^2(t,s)dsdt=\int\limits_{-\infty}^{a}f(t)u(t)dt. \tag{6}$$

Suppose that $f(t)=0$ for $t\in(-\infty,a]$. Then, taking into account conditions (i), (ii) and (iii), we see that (6) implies

$$\int\limits_{-\infty}^{t}u(\tau)d\tau=0,\ t\in(-\infty,a]\ or\ \int\limits_{s}^{a}u(\tau)d\tau=0,\ s\in(-\infty,a]$$

or

$$\int\limits_{s}^{t}u(\tau)d\tau=0,\ (t,s)\in G.$$

Therefore, $u(t)=0$ for all $t\in(-\infty,a]$. The theorem is proved.

Example. Consider the integral equation

$$\int\limits_{-\infty}^{t}H\left(t,s\right)u(s)ds=f(t),\quad t\in(-\infty,0], \tag{7}$$

where

$$H(t,s)=-\frac{c}{a(b-a)}\left[e^{bt}e^{a(t-s)}-\frac{2b}{a+b}e^{as}\right]+\frac{cd}{a+b}(e^{bt}-2),$$
$$(t,s)\in G=\{(t,s);-\infty<s\le t\le 0\}, \tag{8}$$

$a,b,$ c and d are real parameters, $a>0,b>0,c>0,d>0,a\neq b.$ Then from (8) we have

$$H_t^{'}(t,s)=-\frac{c}{a}e^{bt}e^{-a(t-s)}+\frac{cd}{a}e^{bt},\ (t,s)\in G, \tag{9}$$

$$H_s^{'}(t,s)=-\frac{c}{b-a}\left[e^{bt}e^{-a(t-s)}-\frac{2b}{a+b}e^{as}\right],\ (t,s)\in G, \tag{10}$$

$$H_{ts}^{''}(t,s)=-ce^{bt}e^{-a(t-s)},\ (t,s)\in G, \tag{12}$$

$$\alpha(t)=\lim_{s\to-\infty}H(t,s)=0,\ \alpha'(t)=0,\ t\in(-\infty,0], \tag{13}$$

$$\beta(s)=H_s^{'}(0,s)=\frac{c}{a+b}e^{as},\ s\in(-\infty,0]. \tag{14}$$

From (8) and (10), we obtain

$$l\le\frac{c}{|b-a|a}\left(1+\frac{2b}{a+b}\right)+\frac{2c|d|}{ab}, \tag{15}$$

$$l_1 \le \frac{c(a+3b)}{|b-a|\,a(a+b)}, \quad (16)$$

$$\gamma(s) = \frac{c}{|b-a|}\left[e^{bs} + \frac{2b}{a+b}e^{as}\right], s \in (-\infty, 0]. \quad (17)$$

Then taking into account (8)-(17), we can verity that conditions (i), (ii) and (iii) are satisfied for the integral equation (7). Therefore the solution of the integral equation (7) is unique in the space $L_1(-\infty, 0]$.

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MSC 35A25

SOLVING OF QUASILINEAR DIFFERENTIAL EQUATIONS OF THE FIRST ORDER WITH THE CAUCHY CONDITION BY THE METHOD OF ADDITIONAL ARGUMENT

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Investigation using the method of additional argument of first order quasilinear differential equations with the initial Cauchy condition.

Purpose of the article: Using the method of additional argument, reduce quasilinear differential equations of the first order with the initial Cauchy condition to systems of integral equations.

Keywords: additional argument, initial Cauchy condition, quasilinear differential equation, integral equation.

Кошумча аргумент ыкмасын колдонуу менен баштапкы Коши шарты коюлган квазисызыктуу дифференциалдык теңдеменин чыгарылышын изилдөө.

Макаланын максаты: кошумча аргумент ыкмасын колдонуу менен баштапкы Коши шарты койулган квазисызыктуу дифференциалдык теңдеменин чыгарылышын изилдөөнүн негизинде каралган маселени интегралдык теңдемелер системасына келтирүү ыкмасын киргизүү.

Урунттуу сөздөр: кошумча аргумент, баштапкы Коши шарты, квазисызыктуу дифференциалдык теңдеме, интегралдык теңдеме.

Исследование с применением метода дополнительного аргумента квазилинейных дифференциальных уравнений первого порядка с начальным условием Коши.

Цель статьи: С помощью метода дополнительного аргумента свести квазилинейные дифференциальные уравнения первого порядка с начальным условием Коши к системам интегральных уравнений.

Ключевые слова: дополнительный аргумент, начальное условие Коши, квазилинейное дифференциальное уравнение, интегральное уравнение.

In [2, 3] it was shown that by using the method of an additional argument [1] can be explored a system of differential equations with different initial-boundary conditions.

And the following quasilinear partial differential equation

$$\frac{\partial u(t,x,y)}{\partial t}+a_1(t,x,y,u(t,x,y))\frac{\partial u(t,x,y)}{\partial x}+a_2(t,x,y,u(t,x,y))\frac{\partial u(t,x,y)}{\partial y}= \\ = f(t,x,y,u(t,x,y)) \quad (1)$$

with initial conditions:

$$u(0,x,y)=\varphi(x,y) \quad (2)$$

$t\in[0,T]$, $T=const$, $(x,y)\in R^2$ was considered in the paper [4].

It is assumed, that the functions $f(t,x,y,u)$ and $\varphi(x,y)$ are continuously differentiable with respect to all their arguments.

Using the method of additional argument [1, 2], the problem (1) - (2) was reduced to the following system of differential equations:

$$\begin{cases} \dfrac{dp_1(s,t,x,y)}{ds}=a_1(s,p_1(s,t,x,y),p_2(s,t,x,y),w(s,t,x,y)) \\ \dfrac{dp_2(s,t,x,y)}{ds}=a_2(s,p_1(s,t,x,y),p_2(s,t,x,y),w(s,t,x,y)) \\ \dfrac{dw(s,t,x,y)}{ds}=f(s,p_1(s,t,x,y),p_2(s,t,x,y),w(s,t,x,y)) \end{cases} \quad (3)$$

with additional terms:

$$\begin{cases} p_1(t,t,x,y) = x; \\ p_2(t,t,x,y) = y \end{cases} \tag{4}$$

$$w(0,t,x,y) = \varphi(p_1(0,t,x,y), p_2(0,t,x,y)). \tag{5}$$

In this paper, we show that the problem (3)-(5) can be reduced to a system of integral equations and prove their equivalence.

Having integrated the first two equations of system (3) with respect to *s* from *s* to *t*, we obtain, taking into account the conditions (4), two integral equations:

$$p_1(s,t,x,y) = x - \int_s^t a_1(v, p_1(v,t,x,y), p_2(v,t,x,y), w(v,t,x,y))dv,$$

$$p_2(s,t,x,y) = y - \int_s^t a_2(v, p_1(v,t,x,y), p_2(v,t,x,y), w(v,t,x,y))dv. \tag{6}$$

We integrate the last equation of system (3) with respect to *s* from *0* to *s* and, taking into account (5), (6), we obtain the integral equation:

$$w(s,t,x,y) = \varphi\left(\begin{matrix} x - \int_0^t a_1(v, p_1(v,t,x,y), p_2(v,t,x,y), w(v,t,x,y))dv, \\ y - \int_0^t a_2(v, p_1(v,t,x,y), p_2(v,t,x,y), w(v,t,x,y))dv \end{matrix} \right) +$$

$$+ \int_0^s f(v, p_1(v,t,x,y), p_2(v,t,x,y), w(v,t,x,y))dv. \tag{7}$$

So, we have got that the system of integral equations (6), (7) follows from the problem (3)-(5).

By direct differentiation with respect to equations (6), (7), we obtain system (3). Substituting $s=t$ in (6), we obtain the conditions (4), and substituting $s=0$, we obtain the conditions (5).

Thus, the Cauchy problem (3)-(5) and the system of integral equations (6), (7) are completely equivalent.

Now let $u(t,x,y)$ be a solution to the Cauchy problem (1), (2). Having determined $p_1(s,t,x,y)$, $p_2(s,t,x,y)$ from the system of differential equations

$$\frac{\partial p_1(s,t,x,y)}{\partial s} = a_1(s, p_1(s,t,x,y), p_2(s,t,x,y), u(s, p_1(s,t,x,y), p_2(s,t,x,y))),$$

$$\frac{\partial p_2(s,t,x,y)}{\partial s} = a_2(s, p_1(s,t,x,y), p_2(s,t,x,y), u(s, p_1(s,t,x,y), p_2(s,t,x,y)).$$

With the conditions (4), we obtain that the function $w(s,t,x,y) = u(s, p_1(s,t,x,y), p_2(s,t,x,y))$ will satisfy the last equation of the system (3) and the initial condition.

This proves that each solution *u(t, x, y)* to the Cauchy problem (1), (2) gives a solution to the Cauchy problem (3)-(5), and hence does a solution to the system of integral equations (6), (7).

Lemma. Let the functions $(p_1(s,t,x,y),\ p_2(s,t,x,y),\ w(s,t,x,y)) \in C^{1,1,1,1}(R^4)$ and $\Omega_1=\{(s,t,x,y):\ 0 \le s \le t,\ (x,y) \in R^2\}$, then the solution to the system of integral equations (6), (7) gives the solution to the Cauchy problem (1), (2) for $0 < t < T_0 \le T$, where

$$T_0 < \frac{1}{N_f + (1+N_\varphi)(N_{a_1} + N_{a_2})} - \delta,$$

δ is an arbitrarily small number.

Proof. We denote by D^* the differential operator

$$D^* = \frac{\partial^*}{\partial t} + a_1(t,x,y,u)\frac{\partial^*}{\partial x} + a_2(t,x,y,u)\frac{\partial^*}{\partial y},$$

where $u(t,x,y)=w(t,t,x,y)$.

We apply this operator sequentially to all equations of system (6), (7).

We have:

$$Dp_1 = a_1(t,x,y,u) - a_1(t,x,y,u) - \int_s^t [\partial_{p_1} a_1 \cdot Dp_1 + \partial_{p_2} a_1 \cdot Dp_2 + \partial_w a_1 \cdot Dw] dv,$$

$$Dp_1 = -\int_s^t [\partial_{p_1} a_1 \cdot Dp_1 + \partial_{p_2} a_1 \cdot Dp_2 + \partial_w a_1 \cdot Dw] dv. \qquad (9)$$

Similarly, we get

$$Dp_2 = -\int_s^t [\partial_{p_1} a_2 \cdot Dp_1 + \partial_{p_2} a_2 \cdot Dp_2 + \partial_w a_2 \cdot Dw] dv. \qquad (10)$$

Applying the introduced differential operator D^* to equation (7) and taking into account (9), (10) we obtain:

$$Dw = -\partial_x \varphi [-\int_0^s [\partial_{p_1} a_1 \cdot Dp_1 + \partial_{p_2} a_1 \cdot Dp_2 + \partial_w a_1 \cdot Dw] dv] -$$

$$-\partial_y \varphi [-\int_0^s [\partial_{p_1} a_2 \cdot Dp_1 + \partial_{p_2} a_2 \cdot Dp_2 + \partial_w a_2 \cdot Dw] dv] +$$

$$+\int_s^t [\partial_{p_1} f \cdot Dp_1 + \partial_{p_2} f \cdot Dp_2 + \partial_w f \cdot Dw] dv . \quad (11)$$

We denote:

$$N_\varphi = \max\left\{\sup_{R^2} |\varphi|, \sup_{R^2} |\partial_x \varphi|, \sup_{R^2} |\partial_y \varphi|\right\}$$

$$N_{a_i} = \max\left\{\sup_\Omega |a_i|, \sup_\Omega |\partial_{p_1} a_i|, \sup_\Omega |\partial_{p_2} a_i|, \sup_\Omega |\partial_w a_i|\right\},\ i=1,\ 2$$

$$N_f = \max\left\{\sup_\Omega |f|, \sup_\Omega |\partial_{p_1} f|, \sup_\Omega |\partial_{p_2} f|, \sup_\Omega |\partial_w f|\right\}$$

The constants N_{a_1}, N_{a_2}, N_f are determined taking into account the fact that the functions p_1, p_2, w included in a_1, a_2, f are known.

Using the introduced denotations, we obtain from (9), (10), (11):

$$|Dp_i| \le tN_{a_i}(\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega,\ i=1,\ 2,$$

$$\begin{gathered}|Dw| \le tN_\varphi (N_{a_1} + N_{a_2})(\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega) + \\ +tN_f(\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega)\end{gathered}.$$

Since the last inequalities are valid for all $0 \le s \le t$, $(x, y) \in R^2$, it follows from them:

$$\|Dp_i\|_\Omega \le tN_{a_i}(\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega,\ i=1,\ 2,$$

$$\begin{gathered}\|Dw\|_\Omega \le tN_\varphi (N_{a_1} + N_{a_2})(\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega) + \\ +tN_f(\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega)\end{gathered}.$$

Adding these inequalities, we get:

$$\begin{gathered}\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega \le \\ \le t(N_f + (1+N_\varphi)(N_{a_1} + N_{a_2}))(\|Dp_1\|_\Omega + \|Dp_2\|_\Omega + \|Dw\|_\Omega)\end{gathered}.$$

Hence it follows that for $t < \dfrac{1}{N_f + (1+N_\varphi)(N_{a_1} + N_{a_2})}$ the identity

$$\| Dp_1 \|_\Omega + \| Dp_2 \|_\Omega + \| Dw \|_\Omega = 0$$

is true.

We denote $T_0 = \dfrac{1}{N_f + (1+N_\varphi)(N_{a_1} + N_{a_2})} - \delta$, where δ is an arbitrarily small number.

From the above tabs it follows that for $0 < s < T_0 \le T$ the following equalities are true:

$$\frac{\partial p_i}{\partial t} + a_1(t,x,y,u)\frac{\partial p_i}{\partial x} + a_2(t,x,y,u)\frac{\partial p_i}{\partial y} = 0$$

$$\frac{\partial w}{\partial t} + a_1(t,x,y,u)\frac{\partial w}{\partial x} + a_2(t,x,y,u)\frac{\partial w}{\partial y} = 0 . \qquad (12)$$

Substituting now *u (t, x, y) = w (t, t, x, y)* into the equation (1). We will have

$$\frac{\partial w(s,t,x,y)}{\partial s}\Big|_{s=t} + \frac{\partial w(s,t,x,y)}{\partial t}\Big|_{s=t} + a_1\frac{\partial w(t,t,x,y)}{\partial x} + a_2\frac{\partial w(t,t,x,y)}{\partial y} = f .$$

Taking into account the equalities (12) and $\dfrac{\partial w(s,t,x,y)}{\partial s}\Big|_{s=t} = f(t,x,y,u)$ we obtain the correct equality

$$f(t, x, y, u) = f(t, x, y, u).$$

Substituting $s=t=0$ in (7), we obtain

$$u(0, x, y)=w(0,0, x, y)=\varphi(x, y).$$

Thus, it has been proved that the function *w (s, t, x, y)* for *s=t* gives a solution to the initial value problem (1)-(2).

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MSC 34K20, 45J05

SUFFICIENT CONDITIONS OF BOUNDEDNESS OF SOLUTIONS OF ONE NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL SECOND-ORDER EQUATION ON THE HALF-AXIS

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Sufficient conditions of boundedness on the half-axle of all solutions and their first derivatives of a single nonlinear integro-differential equation of the second-order Volterra type are established. For this purpose, a partial cutting method is developed. The illustrative example is given.

Key words: nonlinear integro-differential equation, boundedness, boundedness of solutions, boundedness of first derivatives of solutions, partial slicing method.

Вольтерра тибиндеги экинчи тартиптеги бир сызыктуу эмес интегро-дифференциалдык теңдеменин бардык чыгарылыштарынын жана алардын биринчи туундуларынын жарым окто чектелгендигинин жетиштүү шарттары табылат. Бул үчүн жекече кесүү методу өнүктүрүлөт. Иллюстративдик мисал тургузулат.

Урунттуу сөздөр: сызыктуу эмес интегро-дифференциалдык теңдеме, чектелгендик, чыгарылыштардын чектелгендиги, чыгарылыштардын биринчи туундуларынын чектелгендиги, жекече кесүү методу.

Устанавливаются достаточные условия ограниченности на полуоси всех решений и их первых производных одного нелинейного интегро-дифференциального уравнения второго порядка типа Вольтерра. Для этого развивается метод частичного срезывания. Строится иллюстративный пример.

Ключевые слова: нелинейное интегро-дифференциальное уравнение, ограниченность, ограниченность решений, ограниченность первых производных решений, метод частичного срезывания.

All appearing functions from $t,(t,\tau),(t,x,y)$ are continuous and the relations take place at $t \ge t_0$, $t \ge \tau \ge t_0$; $|x|,|y| < \infty$; $J = [t_0, \infty)$; IDE is a integro-differential equation.

PROBLEM. Establish sufficient conditions of boundedness on the half-interval J for all solutions and their first derivatives of the following nonlinear second-order Volterra type IDE:

$$x''(t) + h(t, x(t), x'(t) + a(t)g(x(t)) + \int_{t_0}^{t} K(t,\tau)\, x'(\tau)d\tau = f(t), \quad t \ge t_0 \tag{1}$$

Note that in [1] such a problem is studied for a more general IDE than (1) the method of weighting and cutting functions [2]. In the present paper, the partial cutting method is developed to solve of the problem formulated above [3,4].

The solution of IDE (1) is the solution $x(t) \in C^2(J, R)$ with any initial data $x^k(t_0)(k = 0,1)$. We will assume the existence of such a solution, although, as noted in [4], it is possible to establish the existence of $x(t) \in C^2(J, R)$ by the method of monotone operators [5].

Let [2-4]:

$$K(t,\tau) = \sum_{i=1}^{n} K_i(t,\tau), \tag{K}$$

$$f(t) = \sum_{i=1}^{n} f_i(t), \tag{f}$$

$\psi_i(t)$ $(i = 1 \ldots n)$ are some cutting functions,

$$P_i(t) \equiv K_i(t,\tau)(\psi_i(t))^{-2}, Q_i(t,\tau) \equiv K_i(t,\tau)(\psi_i(\tau))^{-1},\ E_i(t) \equiv f_i(t)(\psi_i(t))^{-1} \ (i = 1 \ldots n),$$

$$P_i(t) = A_i(t) + B_i(t) \ (i = 1 \ldots n), \tag{P}$$

$c_i(t)(i = 1 \ldots n)$ are some functions.

$Q_i(t,\tau)$ $(i = 1 \ldots n)$ are named partially cut kernels [3].

For an arbitrarily fixed solution $x(t)$ of IDE (1) we multiply by $x'(t)$ [6, p. 194-217], and conducting integrate within the limit from t_0 to t, including in parts, and similarly [3,4] we introduce functions $\psi_i(t), P_i(t), Q_i(t,\tau)$, using lemma [3], if we introduce condition (*P*), functions $c_i(t)\,(i=1\ldots n)$. Then we obtain the following identity:

$$(x'(t))^2+2\int_{t_0}^{t}x'(s)h(s,x(s),x'(s))ds+a(t)G(x(t))+\sum_{i=1}^{n}\{A_i(t)(X_i(t,t_0))^2+B_i(t)(X_i(t,t_0))^2-$$

$$-2E_i(t)X_i(t,t_0)+c_i(t)-\int_{t_0}^{t}[B_i'(s)(X_i(s,t_0))^2-2E_i'(s)X_i(s,t_0)+c_i'(s)]ds\}\equiv$$

$$\equiv c_*+\int_{t_0}^{t}\{a_i'(s)G(x(s))+$$

$$+\sum_{i=1}^{n}[A_i'(s)(X_i(s,t_0))^2+2\int_{t_0}^{s}Q_{i\tau}'(s,\tau)\,X_i(\tau,t_0)\,x'(s)]d\tau\}\,ds, \qquad (2)$$

where $G(x)\equiv\int_0^x g(u)du,\;\; X_i(t,t_0)\equiv\int_{t_0}^{t}\psi_i(\eta)x'(\eta)d\eta \quad (i=1\ldots n),$

$$c_*=(x'(t_0))^2+a(t_0)G(x(t_0))+\sum_{i=1}^{n}c_i(t_0).$$

Turning to the integral inequality from identity (2), applying Lemma 1[7], Theorem 1[1] similarly is proved.

Theorem. Let 1) the conditions are met *(K), (f), (P);*

2) $x'h(t,x,x')\equiv H(t,x,x')\geq 0;$ 3) $a(t)\geq a_0>0,$ the function exists $a_*(t)\in L^1(J,R_+)$ is that $a'(t)\leq a^*(t)\,a(t),\; G(x)\to\infty$ at $|x|\to\infty;$ 4) $A_i(t)>0$, functions exist $A_i^*(t)\in L^1(J,R_+)$ is that $A_i'(t)\leq A_i^*(t)A_i(t)\;(i=1\ldots n);$ 5) $B_i(t)\geq 0,\; B_i'(t)\leq 0,$ functions exist $c_i(t)$ is that $(E_i^{(k)}(t))^2\leq B_i^{(k)}(t)c_i^{(k)}\;(i=1\ldots n;\; k=0,1);$

6) $\int_{t_0}^{t}|Q_{i\tau}'(t,\tau)|(A_i(\tau))^{-\frac{1}{2}}d\tau\in L^1(J,R_+)\quad(i=1\ldots n).$

Then any solution $x(t)$ of the IDE (1) of $C^2(J,R)$ and its first derivative $x'(t)$ are bounded on *J* and the relation is correct:

$$H(t,x,x')\in L^1(J,R_+). \qquad (3)$$

The theorem be obtained a corollary similar to that in [4].

Corollary. If 1) all conditions of the theorem and $H(t,x,x')\geq D(t)(x')^{2m}, m\in N$; 2) $D(t)\geq D_0>0$ (respectively $D(t)>0,\ (D(t))^{-1}\in L^1(J,R_+\setminus\{0\})$ are fulfilled, then for any solution $x(t)\in C^2(J,R)$ of IDE (1) the following properties are true: $x'(t)\in L^{2m}(J,R)$ (respectively $x'(t)\in L^m(J,R))\,(m\in N)$.

The first statement at once follows from (3), and the second - from the relation: $2|x'|^m=2(D(t))^{\frac{1}{2}}|x'|^m.\ (D(t))^{-\frac{1}{2}}\leq D(t)(x')^{2m}+D(t)^{-1}$, by integration within the limit from t_0 to t.

Example. For IDE:

$$x''(t)+(t^2+1)(x'(t))^3+\frac{t+1}{t+2}x^5(t)+\int_0^t e^{t+\tau}\sqrt{e^{-20t}(t-\tau)+25}\,x'(\tau)d\tau=-2e^t,\ t\geq 0$$

all conditions of the theorem and the corollary are fulfilled, here

$$t_0=0,\ H(t,x,x')\equiv(t^2+1)(x')^4,\ a(t)\equiv\frac{t+1}{t+2}\geq\frac{1}{2}=a_0,\ a_*(t)\equiv\frac{1}{(t+1)(t+2)},\ n=1,$$

$$\psi_1(t)\equiv e^t,\ P_1(t)\equiv 5,\ A_1(t)\equiv 3,\ B_1(t)\equiv 2,\ E_1(t)\equiv -2,\ c_1(t)\equiv 2,\ Q_1(t,\tau)\equiv e^t\sqrt{e^{-20t}(t-\tau)+25}.$$

Consequently, all solutions and their first derivatives are bounded at $t\in R_+$, and $x'(t)\in L^4(R_+,R)\cap L^2(R_+,R)$.

Thus, we managed to find a class of IDE of the form (1) for which the above task is solvable.

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MSC 49M37

MODELS OF OVERCOMING STRUCTURAL IMBALANCES OF THE ECONOMY OF THE KYRGYZ REPUBLIC

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One of the most difficult to solve problems of economic reforms in transition economies is to overcome structural imbalances in the economic system inherited from the planned economy. This article is devoted to the analysis and modeling of the process of structural imbalances and the forecast of the development of the economy of the Kyrgyz Republic. Methods of analyzing structural shifts in the national economy using an econometric model based on the production function are proposed. The necessity of developing a three-sector model based on the production function for the analysis and forecasting of economic development is substantiated. The analysis of the state of socio-economic development, investment climate in the Kyrgyz Republic is given.

Keywords: structural changes in the national economy, econometric models, structural disproportion, three-sector model of the economy, production functions, forecast.

Өткөөл экономикалуу өлкөлөрдөгү экономикалык реформалардын чечилбеген кыйла татаал проблемаларынын бири пландуу экономикадан калган экономикалык системадагы түзүлүштүк диспропорцияны жоюу болуп саналат. Бул макала Кыргыз Республикасынын экономикалык өнүгүүсүнүн түзүлүштүк диспропорциялар процессин жоюуу үчүн аны анализдөөгө, моделдөөгө жана өнүгүүсүн прогноздоого арналган. Макалада улуттук экономикадагы түзүлүштүк өзгөрүүнү анализдөө үчүн өндүрүштүк функциянын негизинде түзүлгөн эконометрикалык моделдөө методу сунушталат. Экономиканы өнүктүрүүнү талдоо жана анализдөө үчүн өндүрүштүк функциянын негизинде үч тармактык моделди иштеп чыгуу зарылдыгы негизделген. Кыргыз Республикасынын социалдык-экономикалык өнүгүүсүнүн абалына, инвестициялык климатына талдоо жүргүзүлдү.
Урунттуу сөздөр: улуттук экономиканын түзүлүштүк өзгөрүүлөрү, эконометрикалык моделдер, түзүлүштүк диспропорция, экономиканын үч тармактык модели, өндүрүштүк түзүлүштүк, прогноз.

Одна из наиболее сложно разрешимых проблем экономических реформ в странах с переходной экономикой заключается в преодолении структурных диспропорций в экономической системе, унаследованных от плановой экономики. Данная статья посвящена анализу и моделированию процесса структурных диспропорций и прогнозу развития экономики Кыргызской Республики. Предлагаются методы анализа структурных сдвигов в национальной экономике с помощью эконометрической модели на основе производственной функции. Обоснована необходимость разработки трехсекторной модели на основе производственной функции для анализа и прогнозирования развития экономики. Дан анализ состояния социально-экономического развития, инвестиционного климата в Кыргызской Республике.
Ключевые слова: структурные изменения национальной экономики, эконометрические модели, структурная диспропорция, трехсекторная модель экономики, производственные функции, прогноз.

For the Kyrgyz Republic, as well as other CIS countries, the intractable problem of structural imbalances in the economy inherited from the administrative command system of the USSR is relevant in the context of economic reforms.

The decline in consumption and the growth of capital investments in the country's economy contributed to the preservation of a number of structural imbalances.

The Republic does not have sufficient industrially developed hydropower and natural – geographical resources, which are the basic driving force of economic growth. During the years of independence, the republic was unable to ensure stable economic development, and in terms of the level of poverty of the population is one of the last places in the CIS. Serious problems are associated with the deterioration of industrial and social infrastructure. The gap in the level of development between the city and the countryside, as well as the regions of the country, is deepening.

These problems have been particularly exacerbated by the coronavirus pandemic. According to preliminary data of the NSC of the Kyrgyz Republic, the gross domestic product in January-March 2021 by January-March 2020 amounted to 90.6% [1]. Negative dynamics of economic activity was observed in the republic in the first half of 2020. Against the background of government measures to counter the spread of the coronavirus pandemic and temporary restrictions on the work of economic entities, macroeconomic risks gradually increased from January to April 2020 and inflation accelerated. However, by the end of the first half of 2020 there was a slowdown in the growth of prices, in particular, for food, which is associated with a decline in domestic demand and the consequences of restrictions on business activities. But, despite the decline in economic activity, there was a slight increase in financial intermediation indicators [3, p. 8].

With Kyrgyzstan's entry into the international market, the EAEU required specialists and scientists to develop new methods and models for analyzing and forecasting the development of the country's economy. New methods and models should take into account the macroeconomic intersectoral interrelations of the economy and solve the problem of food security.

The formation of the economic structure of the republic in the years of the USSR caused the emergence of serious imbalances in the period of the market economy. Of particular note are such as the intensive development of raw materials industries, speculative activity in the field of trade and services against the background of the decline of manufacturing industries, etc. The importance of accelerated development, especially of medium- and high-tech manufacturing industries in the country, is recognized at all levels of government. However, in practice, proper modernization processes have not yet been implemented.

For objective and subjective reasons, progressive changes have not occurred, and thirty years later, the state and society are essentially facing the same issues and tasks.

Structural shifts lead to the redistribution of economic resources between sectors of the national economy and can lead to the modernization of the structure of the economy. According to the economic content, structural shifts are reduced to radical qualitative transformations of the interrelations of the elements of the economic system, as well as to a change in the ratio between the country's resource base and the needs of the population. Structural shifts in the national economy may be accompanied by changes in the share of various industries in the country's GDP, as well as the number of people employed in the economy.

Structural shifts can generate certain structural imbalances in the economic system of the state. The structure of the country's economy is characterized by the presence of imbalances and uneven development. In countries with economies in transition, structural imbalances (a low share of industry in public production, and within it a high share of extractive industries, an inefficient structure of foreign trade characterized by a predominance in the import of the final product, and in the export of raw materials, underdevelopment of the infrastructure of trade and services, etc. They are a consequence of the implementation of a long-term economic development strategy focused primarily on maximizing the use of the country's resource potential. The authors of official documents and programs have to reckon with these problems as an objective reality.

In the Kyrgyz Republic as in all post-Soviet countries, the change in the ratio between sectors occurred not only as a result of increased labor productivity in the secondary sector and increased demand for services, but as a result of disintegration processes and new integration processes. The disintegration processes between the post-Soviet countries contributed to the reduction of employment in manufacturing industries, that is, in the secondary sector. Disintegration processes have contributed to deindustrialization, which also differs from processes in countries with developed market economies, where these processes, as a rule, have contributed to the movement of labor into high-tech industries, mainly in the tertiary sector. In deindustrialization occurred not only due to an increase in the service sector, but also due to an increase in the raw materials sector [4, p.60].

Table 1. Main indicators by sectors of the Kyrgyz economy*

Year	Industry and construction			Agriculture and extractive industry			Trade and services		
	GDP	labor	investment in fixed assets	GDP	labor	investment in fixed assets	GDP	labor	investment in fixed assets
2000	27,81	27,83	54,41	36,58	36,62	0,06	28,12	27,62	8,3
2010	25,32	25,4	55,82	19,7	18,79	0,14	42,93	43,32	0,14
2019	33,2	28,47	52,5	14,7	13,9	3,22	49,8	50,01	6,8
2020**	30,4	29,32	48,4	13,5	13,3	2,87	47,9	47,6	4,32

*) according to the Ministry of Economy and Finance of the Kyrgyz Republic [2].
**) preliminary calculation.

Analysis of the economy of recent years shows that investment in fixed assets in all sectors of the economy has different rates of change. The share of GDP of the trade and services sector increased rapidly. Below (Table 1) are the statistical data of the NSC of the Kyrgyz Republic on the main sectors of the economy (as a percentage).

In order to quickly introduce market mechanisms and eliminate government interference, price liberalization was carried out. However, price liberalization at the beginning of market reforms caused a sharp rise in the general price level and a heavy blow to the standard of living of the population, to the financial and monetary systems, and generated new serious imbalances in the structure of the country's economy.

Materials and methods of research

To develop a three-sector model of the economy, the main problem is the availability and availability of statistical data in comparable prices of main production assets (MPA) by sector. To analyze the dynamics of the MPA, we used statistical data on investments in fixed assets by sectors of the economy.

The three-sector models used in the research divide the economy into three sectors: the primary (zero) sector of the economy – extraction of raw materials, energy resources, semi-finished products and other consumable materials, the fund-creating (first) – means of labor (buildings, structures, machinery, equipment, power devices and other investment goods of industrial use), consumer (second) – consumer goods [5, p.42]. Note that all the conclusions made for the two-sector model (total product equals total income, total expenses equal total income, are valid for the three-sector model of the economy. In the three-sector model, total expenditures consist of three components: consumption (C), investment (I) and public procurement (G), and total income is allocated to consumption (C), savings (S) and net taxes (T). The main conditions for applying the three-sector model are:

1. In a certain period of time, the technological level of the country's economy is considered constant, a neoclassical production function is used for the model of the economic system:

$$Y_i = F(K_i, L_i), i = 0,1,2.$$

where $Y_i -$ is the gross output by sector; K_i и $L_i -$ are the volumes of fixed assets and labor resources, respectively, by sector.

Suppose that the total number of people employed in the field of production (L) changes relative to constant growth rates v:

$$v = \frac{L(t+1) - L(0)}{L(t)}$$

The differential equation for a continuous period of time takes the form:

$$\frac{dL}{dt} = vL,$$

where $L(0) = L^0$, solution: $L = L^0 e^{vt}$.

2. The depreciation coefficients of direct material costs α_i and the main production funds of the sectors μ_i are constant.

3. Since we consider cash flows in the scheme of foreign trade turnover, we consider the economy to be closed, therefore foreign trade is not considered.

Then, for fixed assets, the differential equation for sectors has the form:

$$\frac{dK_i}{dt} = -\mu_i K_i + I_i; K_i(0) = K_i^0,$$

where i – is the sector number.

Thus, we will write the three-sector model of the economy in the form:

$$\begin{cases} L = L^0 e^{vt} \text{ is total number of employed;} \\ L = L_0 + L_1 + L_2 \text{ is distribution of employed by sector;} \\ \frac{dK_i}{dt} = -\mu_i K_i + I_i; \text{ where } K_i(0) = K_i^0 \text{ is dynamics of funds by sector;} \\ Y_i = F(K_i, L_i), \text{ where } i = 0,1,2 \text{ is output by sector;} \\ Y_1 = I_0 + I_1 + I_2 \text{ is distribution of products of the fund} - \text{creating sector;} \\ Y_2 = \alpha_0 Y_0 + \alpha_1 Y_1 + \alpha_2 Y_2 \text{ is distribution of products of the material sector} \end{cases}$$

Parameters of the three-sector model of the economy:

Y_0 is "Production material costs";

Y_1 is Indicator "Accumulation" minus "Production of consumer goods";

Y_2 is "Non-productive consumption";

K_i is were determined by the indicators "Volume of investment in fixed assets by industry";

L_i is were determined by the indicators "Distribution of the population employed in the economy by industry".

The model system has linear dynamic elements K_i, L_i:

$$\frac{dK_i}{dt} = -\mu_i K_i + I_i \text{ and } \frac{dL}{dt} = vL.$$

Since gross output by sector is set by nonlinear functions, the model system is nonlinear and multiphase, since it contains three variables K_0, K_1 and K_2 interconnected by means of balances.

With the help of the distribution of labor $L = L_0 + L_1 + L_2$ and investment $I_0 + I_1 + I_2 = Y_1$ resources, it is possible to manage economic processes.

To optimize structural changes, the balance sheets of income and expenses by sector were used:

1) The balance of income and expenses of the material sector:

$$\rho_0(1 - \alpha_0)Y_0 = \rho_1 I_0 + t_0 Y_0 + L_0 w_0.$$

2) Balance of income and expenses of the fund-creating sector:

$$\rho_1 Y_1 = \rho_0 \alpha_1 Y_1 + t_1 Y_1 + L_1 w_1.$$

3) The balance of income and expenses of the consumer sector:

$$\rho_2 Y_2 = \rho_0 \alpha_2 Y_1 + \rho_1 I_1 + t_2 Y_2 + L_2 w_2.$$

To calculate the volume of output by sectors, we used the Cobb –Douglas production function:

$$Y = A_i K_i^{\alpha_i} L_i^{1-\alpha_i}, \qquad i = 0,1,2.$$

A_i is the coefficient of neutral technical progress; α_i is the coefficient of elasticity for funds; $1 - \alpha_i$ is the coefficient of elasticity for labor.

The calculations are based on statistics from 2000-2020.

The following PF sectors were obtained:

$$\begin{cases} Y_0 = 1{,}32 K_0^{0{,}42} L_0^{0{,}58} \\ Y_1 = 3{,}41 K_1^{0{,}52} L_1^{0{,}48} \\ Y_2 = 1{,}39 K_2^{0{,}61} L_2^{0{,}39} \\ Y_2 = 1{,}39 K_2^{0{,}61} L_2^{0{,}39} \end{cases} \qquad (1)$$

The main characteristics of the parameters of the production function by sector are given in Table 2.

From the definition of the parameters of the production function by sector (α_0, α_1 и α_2), it follows: an increase in the MPA of raw materials industries by 1% leads to an increase in output on average by α_0%. The same increase in MPA in the fund-creating and consumer industries leads to an increase in output by α_1% and α_2%."

Table 2. The main characteristics of the parameters of the production function

Criteria	Indicators		
	X	*K*	*L*
Industry and construction			
R^2 – is the coefficient of determination	0,77	0,89	0,71
F – Fischer's criterion	37,24	57,13	8,91
Agriculture and extractive industry			
R^2 – is the coefficient of determination	0,87	0,93	0,45
F – Fischer's criterion	41,32	59,25	4,23
Trade and services			
R^2 – is the coefficient of determination	0,82	0,93	0,45
F – Fischer's criterion	33,29	41,33	8,17

Based on the obtained model, the forecast of the studied indicators for the period 2020-2030 is calculated. The forecast values of the parameters show that the sectoral structure of the country's economy will change insignificantly (Table 3). This requires significant changes in the structure in favor of manufacturing industries. For a qualitative change in the structure, it is necessary to optimize the distribution of investments, that is, to create a favorable investment climate. The received foreign direct investments should be directed to the industries of industry and construction, agriculture to create modern production technology and infrastructure, and not to support the old production technology.

Table 3. Share of GDP by sectors of the Kyrgyz economy

Year	Industry and construction	Agriculture and extractive industry	Trade and services
2020*	30,4	13,5	47,9
2021	32,2	13,2	48,5
2022	33,7	13,1	49,6
2023	34,5	12,9	49,8

*According to the NSC KR [1].

Conclusion

During the transition period of economic development, the market mechanism cannot provide the necessary proportion of the distribution of resources of the national economy by sectors, since the basis for the formation of this mechanism is profit.

The analysis of structural shifts shows that for the sustainable development of the national economy, the main goal should be to optimize the intersectoral and intra-sectoral distribution of available resources.

The decision on the priority of investments in various sectors should be made and adjusted by the government both through regulatory methods and through direct participation in the economic process.

The forecast data show that the Government of the country, when developing strategic plans, needs to take into account the existence of economic imbalances at the macro level and in the future it is necessary to coordinate the distribution of foreign investments at the state level.

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MSC 97D40

USE OF DIGITAL TECHNOLOGIES IN THE EDUCATIONAL PROCESS OF HIGHER SCHOOL

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The article discusses the features of e-education and types of information learning tools in the form of block diagrams; the unresolved problems of e-education are shown, the advantages and disadvantages of this education are considered, and a method for resolving these problems is proposed. When using digital technologies as a necessary addition to traditional higher education education, it is required to use educational sites. Examples of these sites are given.
Keywords: digital technologies, information space, e-culture, educational process, professional retraining of teachers, e-education, e-learning, digital educational technologies, information learning tools, educational portal.

Макалада электрондук билим берүүнүн өзгөчөлүктөрү жана маалыматтык окуу куралдарынын түрлөрү блок-схема түрүндө каралат; электрондук билим берүүнүн чечилбеген көйгөйлөрү көрсөтүлүп, бул билим берүүнүн артыкчылыктары жана кемчиликтери каралып, бул көйгөйлөрдү чечүүнүн ыкмасы сунушталат. Санариптик технологияларды салттуу жогорку билим берүүгө зарыл кошумча катары колдонууда билим берүүчү сайттарды колдонуу талап кылынат. Бул сайттардын мисалдары келтирилген.
Урунттуу сөздөр: санариптик технологиялар, маалымат мейкиндиги, электрондук маданият, билим берүү процесси, мугалимдерди кесиптик кайра даярдоо, электрондук билим берүү, санариптик билим берүү технологиялары, маалыматтык окуу куралдары, билим берүү порталы.

В статье рассмотрены особенности электронного образования и виды информационных средств обучения в виде блок-схем; показаны неразрешенные проблемы электронного образования, рассмотрены достоинства и недостатки данного образования и предложен способ разрешения данных проблем. При использовании цифровых технологий в качестве необходимого дополнения к традиционному образованию в высшей школе требуется использовать образовательные сайты. Приведены примеры данных сайтов.

Ключевые слова: цифровые технологии, информационное пространство, электронная культура, образовательный процесс, профессиональная переподготовка преподавателей, электронное образование, электронное обучение, цифровые образовательные технологии, информационные средства обучения, образовательный портал.

The change in the social order for the preparation of competitive specialists for mastering modern media and the ability for self-learning and self-development, and then realizing their creative potential in future practical activities is due to the transformation of modern civilization into an information space that requires intensive formation of information. On no graphic and electronic culture of specialists. Graduates are required not only to have fundamental basic training, which will help them understand complex production, but also information technology readiness, namely: knowledge of information and digital technologies and the ability to handle them; the ability to collect, evaluate and use information; high adaptability in the ability to adapt to new working conditions; communication and ability to work in a group; the ability for self education and the need for regular professional development, etc.

In modern society, with constantly changing socio-economic conditions and the use of digital technologies, the requirements for higher education and graduates have changed:

the qualifications of teachers;

new generation techniques.

The relevance of this article is the low level of knowledge of teachers regarding the use of digital technologies as a teaching tool. In line with the 2020 year learning vision, higher education teachers must undergo elearning vocational training.

In the age of digital technologies, there is a significant need for retraining of teachers of higher education. There is a big gap between the knowledge of teachers working with the old store of knowledge and those using digital technologies, which are constantly increasing in quantity and quality every day. Teachers do not have time to keep track of modern trends in information technology in an avalanche of information due to the heavy workload in their daily work. Since, in addition to the fact that they need to conduct classroom sessions in the first half of the day, which for many lasts more than half a day, in the second half of the day, taking into account new trends coming from the Ministry of Education and Science of the Kyrgyz Republic, taking into account the requirements of the constantly changing federal state educational standards with their competencies, teachers are required to develop educational methodological complexes (TMC), which must be tested in the educational process.

Taking this into account, it is necessary to properly organize the phased retraining of teaching staff in the use of digital technologies and the development of new generation techniques.

At the university, it is necessary to enter an electronic journal, with which not only teachers, but also students and parents should work, in which you can view the electronic schedule, marks and assignments, and also use messages to correspond with teachers, thus forming the electronic culture of the user

After mastering the first stage, there is a need to use technical teaching aids in the field of education development, namely the use of technical teaching aids (TCO) in the educational process in the form of creating information management systems (IMS) for managing the content of electronic documents (ED), verification, homework tests, tests, slices of students' knowledge. There is an opportunity in the form of feedback to view the report of each student on the Internet.

With the use of modern electronic learning technologies, it is possible to personally develop students, taking into account their psychological and physiological abilities. Consider the features of e-education (Figure 1), consisting of 6 modules, namely:

• reducing the time for developing technical skills of students;
• increasing the number of training tasks;
• achievement of the optimal pace of work of the student;

• easily achieved level differentiation of training;

• the student becomes the subject of learning, as he actively has to work in the lesson;

• increasing the motivation of learning activities.

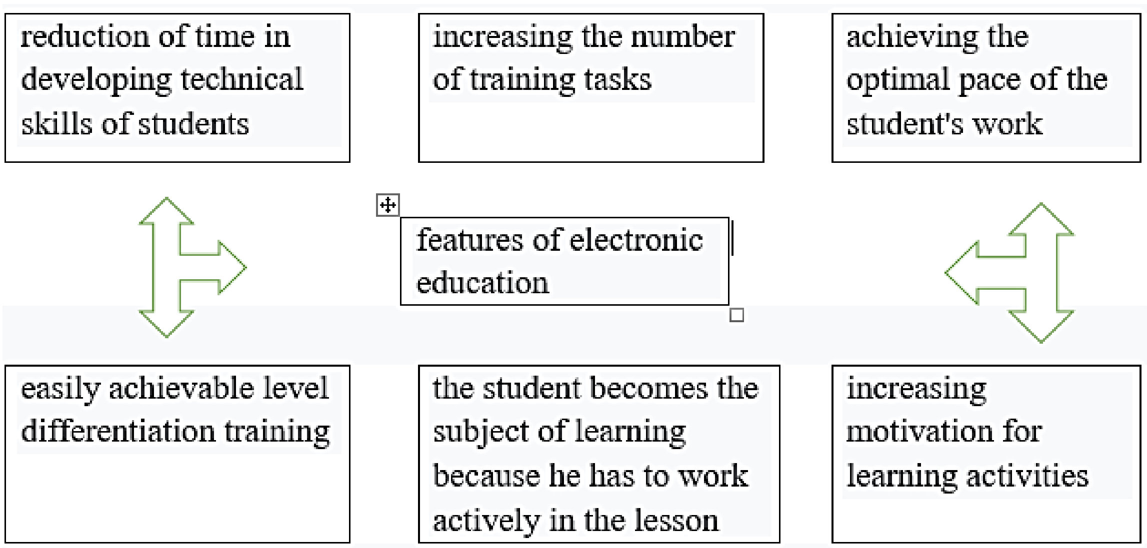

Fig. 1. Block diagram "Features of e-education"

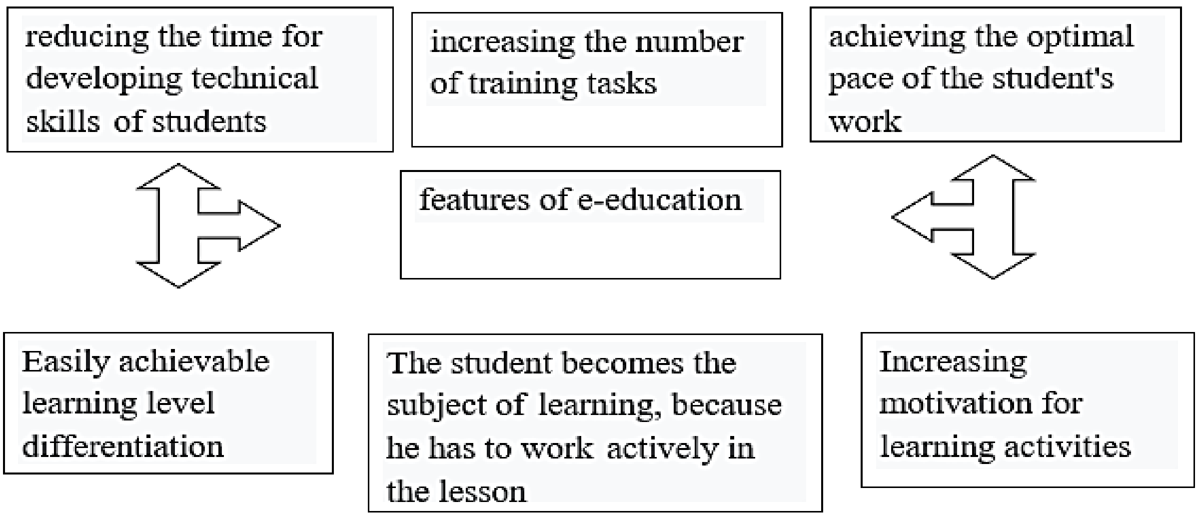

Fig. 2. Block diagram "Types of information training tools for ISO"

From the block diagram

"Features of e-education" (Figure 1) shows that e-education allows you to optimize the pace of work of students and increases the motivation of learning activities using information learning tools (ITS). Next, consider the types of information learning tools (Figure 2), consisting of the following modules:

• IUS, to which the students are connected;

• electronic board (replacing the chalk board);

• electronic journals (scientific, popular science, methodological, art, general education journals in addition to the textbook);

• video conferencing, webinars, etc.

Advantages of e-education	Disadvantages of e-education
Save time	The problem of the quality of e-courses
Flexibility	Legal problems for the protection of intellectual property
Ease of returning to the completed training material	Financial concerns regarding the cost of preparing and updating e-courses
	Staffing problems in training teachers who will be able to develop and update e-courses

To date, electronic education using digital technologies has unresolved problems (Table 1) associated with undeveloped uniform criteria for assessing the quality of electronic disciplines, the composition of competent specialists assessing the quality of these disciplines [2]. To solve the problems of e-education and the effective use of digital technologies in the educational process of higher education, it is necessary to create a single interuniversity center that performs the following functions:

• development of unified requirements for assessing the quality of electronic disciplines;

• training of trainers in e-learning;

• flexible development of standards and competencies in electronic disciplines;

• development of new generation techniques;

• cooperation with IT companies in order to resolve the problems of complex informatization of universities [3].

When using digital technologies as a necessary addition to traditional education in higher education, it is required to use educational sites that have a large base of educational materials necessary for teachers and students in their work, allowing:

• make a presentation of educational materials;

• conduct frontal polls in the group;

• conduct training on the topics of the university program.

Advantages of e-learning Disadvantages of e-education Time saving Problem of quality of e-courses Flexibility Legal problems for the protection of intellectual property Ease of returning to the completed course material Financial problems concerning the costs of preparing and updating e-courses Staff problems for training teachers who will be able to develop and update e-courses Figure 2. Block diagram "Types of informational training tools for ISO" Table 1.

We must not forget that e-learning should not completely replace traditional teaching, it should complement it, since no one and nothing can replace live communication between a teacher and students. The teacher, having feedback with the student, can restructure the teaching material in the course of teaching, making it more understandable and accessible. E-learning itself cannot adapt to the student, since it is controlled by a living mind, in this case the teacher.

Conclusions

The article discusses the features of electronic education and types of educational information tools in the form of block diagrams; the unresolved problems of e-education are shown, the advantages and disadvantages of this education are considered, and a method for solving these problems is proposed. When using digital technologies as a necessary addition to traditional education in higher education, it is required to use educational sites. Examples of these sites are given.

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Формат 60×48. Объем 8,0 п.л.
Подписано в печать 6.12.21г.
Тираж 100 экз.
Отпечатано в ОсОО “Ала-Тоо Полиграф Сервис”
Пр. Чуй 265а